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Biased randomly trapped random walks and applications to random walks on Galton-Watson trees

by

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Contents

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Declarations

I declare that, to the best of my knowledge, the material in this thesis is original and my own work, conducted under the supervision of David Croydon, except otherwise indicated. Notably,

i) Chapter 3 is formed of work from Bowditch [24];

ii) Chapter 4 is formed of work from Bowditch [25];

iii) Chapter 6 contains work from two articles: Section 6.1 is formed from Bowditch [26] and Section 6.2 is formed of work from Bowditch [25];

iv) Chapter 2 contains a range of results which appear in the articles noted above.

The material in this thesis is submitted to the University of Warwick for the degree of Doctor of Philosophy and has not been submitted to any other university or for any other degree.

Abstract

In this thesis we study biased randomly trapped random walks. As our main motivation, we apply these results to biased walks on subcritical Galton-Watson trees conditioned to survive. This application was initially considered model in its own right.

We prove conditions under which the biased randomly trapped random walk is ballistic, satisfies an annealed invariance principle and a quenched central limit theorem with environment dependent centring. We also study the regime in which the walk is sub-ballistic; in this case we prove convergence to a stable subordinator. Furthermore, we study the fluctuations of the walk in the ballistic but sub-diffusive regime. In this setting we show that the walk can be properly centred and rescaled so that it converges to a stable process.

The biased random walk on the subcritical GW-tree conditioned to survive fits suitably into the randomly trapped random walk model; however, due to a lattice effect, we cannot obtain such strong limiting results. We prove conditions under which the walk is ballistic, satisfies an annealed invariance principle and a quenched central limit theorem with environment dependent centring. In these cases the trapping is weak enough that the lattice effect does not have an influence; however, in the sub-ballistic regime it is only possible to obtain converge along specific subsequences.

We also study biased random walks on infinite supercritical GW-trees with leaves. In this setting we determine critical upper and lower bounds on the bias such that the walk satisfies a quenched invariance principle.

Chapter 1

Introduction

Over the last forty years random motions in random media have been intensively studied, resulting in the emergence of many new interesting phenomena, mathematical models and probabilistic techniques. A large driving force in this work has been due to a variety of probabilistic models originating from physical sciences including condensed matter physics, reaction kinetics and polymer dynamics where diffusion in inhomogeneous media is of considerable interest (e.g. [12]). Specifically, models of random traps are of particular interest in physical chemistry where many of the issues are closely linked to the analysis of the Schrödinger equation and random potentials (see [39]).

A fundamental topic in the field of random walk in random environment is the evolution of the displacement of the walk from its origin and how it is influenced by fluctuations in the environment. In a wide range of models of random walks in random environments this is driven by a trapping mechanism where the randomness of the environment creates adverse regions which slow the walk. In recent years there has been much progress in models which involve trapping; a review of recent developments in a range of models of directionally transient and reversible random walks on underlying graphs is given in [9].

As our main example we consider biased random walks on Galton-Watson (GW) trees conditioned to survive. These trees consist of an infinite backbone with finite trees attached as branches. The branches form dead-ends in the environment which makes it a natural setting for observing trapping as the walk is slowed by taking excursions in the finite sections of the tree. The influence of the bias on the trapping is an important feature of the model; as the bias is increased the local drift away from the root will increase but this does not necessarily speed up the walk. This is because it increases the time trapped in the finite leaves from which the walk cannot escape without taking long sequences of movements against the bias.

One of the canonical models in the field is the so-called random walk in random environment. This is not the main focus of the thesis so we will not introduce it in great length; see [74] and [77] for a more detailed history. In this model we consider a fixed graph and sample an environment by randomly choosing the transition probabilities at each vertex independently from a fixed distribution. Trapping is caused by 'bad pockets' in the environment which reverse the imposed drift thus slowing the walk. Although some progress has been made in the generalised set-up (e.g. [47]), most work has focussed around using $\mathbb{Z}^d$ (for $d \geq 1$) as the fixed underlying graph.

The one dimensional model received much attention in the 1970s; firstly in [70] where conditions for transience and an expression for a limiting speed were determined and then in [50] where fluctuations and central limit theorems have been studied in greater detail. There has also been a slightly more recent interest with large deviation results having been determined in [30] and the case of the non-i.i.d. environment being considered in [3]. The renowned 'environment viewed from the particle technique' (introduced in [64] and developed in [29], [51]) acted as a major breakthrough for this model and although the technique has been developed in a more general setting (see [68]) it has had little impact for higher dimensions.

More recently, the higher dimensional model has been studied in greater detail. By using a renewal structure, a law of large numbers has been shown in [75] and, using a then new technique, functional central limit theorems have been proved in [21]. This latter technique allows an annealed invariance principle to be extended to a quenched version when the dimension is sufficiently high. The technique is now sufficiently well developed to be applied in a wide range of random walk models.

Another archetypal model in the field is that of random walk on supercritical percolation clusters. In this model we consider, for the environment, the unique infinite cluster in supercritical Bernoulli bond percolation on $\mathbb{Z}^d$ for $d \geq 2$ (see [41] for an overview). This defines a random graph which, since there is positive probability that the cluster contains any fixed vertex, can be conditioned to contain the origin as a root for the random walk. This is one of many natural examples which exhibit anomalous behaviours caused by inhomogeneity in the environment.

One of the major features of the model is the occurrence of trapping; if we introduce a weak bias then the walk moves at positive speed whereas if the bias is strong then the speed vanishes (see [73]). This rather counter-intuitive behaviour occurs because, as the walker is pushed into new regions by the bias, it encounters dead-ends that act as traps which hinder its escape from the root. The trapping mechanism becomes stronger as the bias is increased because the greater bias makes it more difficult for the walk to escape the traps. This behaviour is very similar to that of the random walk on GW-trees.

Both the isotropic and anisotropic walks have been studied in some detail with renormalisation and harmonic deformation techniques yielding rewarding outcomes. It has been shown in [42] that the isotropic walk is almost surely transient in dimension

$d \geq 3$ (but is centred and therefore clearly sub-ballistic). The anisotropic walk is also transient in dimension $d \geq 2$ (see [14] or [73]) and experiences a phase transition from ballistic to sub-ballistic as the bias is increased above the critical value which has been identified in [37]. Moreover, regimes (both isotropic and anisotropic) such that a functional central limit theorem holds have been determined in [13], [37] and [58].

The Bouchaud trap model was introduced in [22] in order to study aging phenomenon in spin-glasses at low temperature. In this trapping model we randomly assign a depth ω_x to each vertex x of a graph; this forms the environment. For a fixed environment, we then consider a continuous time random walk with independent exponentially distributed holding times with mean ω_x from site x. This model has been discussed in great detail in the physics literature (e.g. [23], [61] and [69]) in the context of non-equilibrium phenomena in disordered systems.

A key feature in many of these physical models is aging in which the decorrelation properties of the system are time dependent. This property of aging relates to localisation of the random walk in the Bouchaud trap model. Indeed, it has been shown in [35] that, for the Bouchaud trap model on $\mathbb{Z}$, if the depths belong to the domain of attraction of a stable law with index $\alpha < 1$ (so that the depths ω_x have infinite mean) then the walk is subdiffusive and (suitably scaled) converges to the FIN singular diffusion. This limit process is a Brownian motion time changed by the inverse of a stable subordinator evaluated at the local time of the Brownian motion (driven by a speed measure associated with the environment). That is, the convergence is such that the limit is environment dependent as the walk is slowed in areas of the graph with particularly deep traps.

The picture in higher dimensions is rather different. In this case we see, as the limit, the fractional kinetics process (see [5], [8] and [63]). That is, a Brownian motion time changed by the inverse of a stable subordinator which is independent of the Brownian motion. Specifically, the walk is slowed to the same extent but the spatial aspect is insignificant because the walk never stays in one area for very long. This is, in fact, the same limit observed for the continuous time random walk with infinite mean waiting time (see [59] and [62]). For a more detailed account we direct the reader to [7] which gives a summary of mathematical results for the Bouchaud trap model in dimensions $d \geq 1$ and also more detail in its relation to spin-glasses.

Recently, a more general model of randomly trapped random walks was introduced in [6] to generalise models such as the Bouchaud trap model and provide a framework for studying random walks on other random graphs in which trapping naturally occurs such as biased random walks on percolation clusters and random walk in random environment. In this general model, rather than defining the trap at each vertex by a single variable (i.e. the depth), we randomly assign to each vertex a probability measure supported on $(0, \infty)$. We consider this collection of probability measures as the environment and then, for a fixed environment, consider a random walk with independent holding times distributed according to the measure associated with the site at which the walk is positioned.

In the seminal paper [6] it is shown that the possible scaling limits of the unbiased randomly trapped random walk on $\mathbb{Z}$ belong to a certain class of time changed Brownian motions called randomly trapped Brownian motions. This class of processes includes both the fractional kinetics process and the FIN diffusion but also a much larger class of processes called spatially subordinated Brownian motions in which the time change encodes the spatial inhomogeneity in a more intricate way than for the FIN diffusion. Higher dimensional ($d \geq 2$) unbiased randomly trapped random walks have been studied further in [27] where a complete classification of the possible scaling limits is given.

These random walk models are just a few of many instances of statistical mechanics in random media that are considered by physicists and mathematicians alike. Many of these act as a stepping stone for the understanding of more complicated processes. For an overview of recent development for some such processes, including the random walk on the incipient infinite cluster and other diffusion processes on fractals, we direct the reader to [52].

The remainder of this chapter will be used to introduce the main models we study throughout the thesis in greater detail. These models include the randomly trapped random walk and biased random walks on subcritical and supercritical GW-trees conditioned to survive. In Chapter 5 we will briefly consider several other models including the Bouchaud trap model. These will be used to demonstrate how the randomness in the model influences the limiting behaviour and to conjecture results for random walks on GW-trees. Because we view these models as a mechanism for studying random walks on GW-trees, we will not describe them in greater detail here, leaving a more precise definition for Chapter 5. A slightly more detailed description of the results we prove for each model is given in Sections 1.1 and 1.2 which describe the models; however, the precise statements of the theorems will always be given at the beginning of the chapter. Throughout the thesis we will use many results for random walks on fixed trees, random walks on $\mathbb{Z}$, properties of stable laws and various other classical results. To avoid repetition and highlight some of the most important results, we will state and prove a range of technical lemmas in Chapter 2.

In Chapter 3 we investigate biased RTRWs on $\mathbb{Z}$ and apply the results to random walks on subcritical GW-trees conditioned to survive. To begin, we prove conditions under which the random walk is ballistic; that is, the walk has a positive limiting speed. We then show that this speed satisfies an Einstein relation; more specifically, that the derivative of the speed (with respect to the bias) converges to half of the diffusion coefficient as we approach the unbiased case. This speed is then used to centre the random walk in order to prove an annealed functional central limit theorem; namely, we consider a renewal argument similar to that of [72] to prove that the position of the walk can be centred and rescaled so that the process converges in distribution to a Brownian motion. We then adapt a technique used in [40] to derive a quenched central limit theorem with an environment dependent centring. We conclude the chapter by applying these results to biased random walks on subcritical GW-trees conditioned to survive.

In Chapter 4 we study biased random walks on subcritical GW-trees conditioned to survive in the sub-ballistic regime. Briefly, the sub-ballistic regimes splits into four phases depending on the bias and the stability of the offspring law. We will discuss these phases in greater detail after a more precise definition of the tree. In one of the phases we have that the walk is recurrent and we do not study this further. In the other three sub-ballistic phases the walk is transient but slowed due to characteristics of the tree. In each of these cases we determine the correct polynomial scaling such that the walker's distance from the root converges to some non-trivial limit.

In Chapter 5 we consider the randomly trapped random walk in the sub-diffusive regime; that is, when the trapping is too strong to obtain a central limit theorem. This splits into two distinct phases depending on whether the holding times have finite mean. When the expected holding time is infinite we have that the walk is sub-ballistic. In this general setting we give conditions under which the position of the walk converges, after suitable rescaling, to the inverse of a stable subordinator. This is not possible for the random walk on the subcritical GW-tree because of a so-called lattice effect which we will explain in greater detail in Section 1.2.2. The other phase we consider is where the holding times have finite mean but infinite variance. In this setting we are able to apply the speed result from Chapter 3 but the fluctuations are too large to obtain a central limit theorem. Here, we prove a condition which shows that, under suitable centring and rescaling, the walk converges in distribution to a stable process. We then apply these results to some of the classical models of random walks in one-dimensional random environments; most notably the comb model which can be seen as a logical intermediary for the study of random walks on subcritical GW-trees.

In Chapter 6 we move onto random walks on supercritical GW-trees conditioned to survive. Unlike the random walk on the subcritical GW-tree conditioned to survive, this model does not easily fit into the randomly trapped random walk model. Despite this, we are able to use many estimates for trapping times proved throughout the thesis alongside existing framework (due to [10], [55] and [65] among others) in order to prove new results. In particular, we extend a result from [65] to prove a quenched functional central limit theorem and develop a result from [10] concerning the correct polynomial scaling of the walk in the sub-ballistic regime.

Throughout the thesis we have used a large amount of notation, some of which varies between chapters. We include a note on the notation used and a list of the important, frequently used notation in the glossary.

1.1 Randomly trapped random walks

We next introduce the general RTRW model in more detail. Fix a graph $G = G(V, E)$ and let $(Y_k)_{k\geq 0}$ be a random walk on G with a law P. For $x \in V$ write

$$\mathcal{L}(x,n) := \sum_{k=0}^{n} \mathbf{1}_{\{Y_k=x\}}$$

for the local time of Y at site x by time n. We define a random environment ω as a sequence of $(0,\infty)$-valued probability measures $(\omega_x)_{x\in V}$ with environment law $\mathbf{P} := \pi^{\otimes V}$ for a fixed law π. For a fixed environment ω, let $(\eta_{x,i})_{x\in V, i\geq 1}$ be independent with $\eta_{x,i} \sim \omega_x$. Writing

$$S_n := \sum_{x\in V} \sum_{i=1}^{\mathcal{L}(x,n-1)} \eta_{x,i} = \sum_{k=0}^{n-1} \eta_{Y_k,\mathcal{L}(Y_k,k)} \qquad \text{and} \qquad S_t^{-1} := \inf\{k \geq 0 : S_k > t\}$$

we then define the randomly trapped random walk by

$$X_t := Y_{S_t^{-1}}.$$

This process is then a continuous time random walk on G with k^{th} holding time $\eta_k := \eta_{Y_k,\mathcal{L}(Y_k,k)}$ and we write $\eta := (\eta_k)_{k\geq 0}$ to be the sequence of holding times. That is, for a fixed environment ω and walk Y, the random variables η_k for $k \geq 0$ are independent with law ω_{Y_k}. For convenience we will define $S_t = S_{\lfloor t\rfloor}$ where $\lfloor t\rfloor := \max\{k \in \mathbb{Z} : k \leq t\}$ for non-integer $t \in \mathbb{R}$. We let P^ω denote the law over X for the fixed environment ω and $\mathbb{P}(\cdot) := \int P^\omega(\cdot)\mathbf{P}(\mathrm{d}\omega)$ the annealed law.

This model was first introduced in [6] primarily to study the case in which the embedded walk $(Y_k)_{k\geq 0}$ is a simple, symmetric random walk on $\mathbb{Z}$. This setting is used to develop the foundations and gain some understanding of the possible scaling limits that arise in this general model. It is shown that the scaling limits belong to a large class of time changed Brownian motions (called randomly trapped Brownian motions) where the time change may retain much of the randomness of the spatial inhomogeneity. This includes the FK-process, the FIN diffusion and also a new class of processes called spatially subordinated Brownian motions which are time changes of Brownian motions where the time change reflects the randomness of the spatial structure in a more complex manner than in the FIN case. Specifically, it is shown that the asymptotic behaviour of X_t is, in general, a mixture of a fractional kinetics process and a spatially subordinated Brownian motion.

In [27], the model in which the embedded walk is a genuinely d-dimensional centred random walk on $\mathbb{Z}^d$ (for $d \geq 2$) with finite range jump distribution is studied. In this setting, it is shown that the only scaling limits are the constant time changed Brownian motion and the fractional kinetics process. That is, the limiting process is independent of the spatial randomness because the embedded walk does not spend a significant amount of time in any area of the graph. This generalises previously well known results for models such as the continuous time random walk [59] and the Bouchaud trap model [8]. These results for the general model of randomly trapped random walk are in the annealed setting whereas, in many cases, quenched results are known for specific models; for instance, the Bouchaud trap model [36], [63]. It should be the case that the annealed convergence to Brownian motion extends to a quenched result in high enough dimension by using the technique developed in [21]. The reason for this is that the randomness of the embedded walk creates a mixing in the environment which results in no specific trap (or 'small' collection of traps) having a significant impact on the fluctuations. We consider this approach later when we investigate random walks on supercritical GW-trees.

In this thesis we consider the randomly trapped random walk model in which the embedded walk $(Y_k)_{k\geq 0}$ is a simple, biased random walk on $\mathbb{Z}$. That is, we write $Y_k := \sum_{j=1}^k \chi_j$ for a sequence of i.i.d. random variables $(\chi_j)_{j\geq 1}$ satisfying $P(\chi_j = -1) = (\beta+1)^{-1} = 1 - P(\chi_j = 1)$. This model will be the main focus of Chapters 3 and 5. Although we have the same underlying graph as studied in [6], we do not observe time changes which retain the randomness of the spatial composition in the same way. The reason for this is that the bias constantly forces the walk into new regions of the graph making it unlikely that the walk spends a large amount of time in any finite region. The walk is, therefore, transient and in many ways behaves somewhat similarly to the walk in a higher dimensional graph. The main exception to this is that the walk can only escape along a single path; this means that, in the quenched setting, the fluctuations of the walk are, in part, driven by the specific inhomogeneity in the environment.

In Chapter 3 we prove a law of large numbers and two central limit theorems for the randomly trapped random walk on $\mathbb{Z}$. That is, in Theorem 3.1, we show that if the bias is positive ($\beta > 1$) and the expected holding time η_0 (under $\mathbb{P}$) is finite, then X_{nt}/n converges $\mathbb{P}$-a.s. to the deterministic process $\nu_\beta t$ where ν_β is a known constant (called the speed). We then show that this speed satisfies an Einstein relation; more specifically,

$$\lim_{\beta\to 1^+} \frac{\mathrm{d}}{\mathrm{d}\beta}\nu_\beta = \frac{\Upsilon}{2}$$

where Υ is the diffusion coefficient of the unbiased walk.

Following this, we prove the annealed invariance principle Theorem 3.2. This states that, if $\beta > 1$ and $\mathbb{E}[\eta_0^2] < \infty$ then for some $\varsigma \in (0,\infty)$

$$B_t^n := \frac{X_{nt} - nt\nu_\beta}{\varsigma\sqrt{n}}$$

converges in $\mathbb{P}$-distribution to a Brownian motion. We prove this by considering a renewal argument similar to that of [72].

For the final CLT result for the RTRW, we adapt the technique used in [40] (to prove a quenched CLT for a random walk in random environment) to derive a quenched central limit theorem with an environment dependent centring (Theorem 3.3). That is, we show that if $\beta > 1$, $\mathbb{E}[\eta_0^2] < \infty$ and for some $\varepsilon > 0$ that $\mathbf{E}[E^\omega[\eta_0]^{2+\varepsilon}] < \infty$ then there exists $\vartheta \in (0,\infty)$ such that for $\mathbf{P}$-a.e. ω there exists an environment dependent centring $\mathcal{G}^\omega(n)$ such that

$$\frac{X_n - \mathcal{G}^\omega(n)}{\vartheta\sqrt{n}}$$

converges in distribution to a standard Gaussian. In particular, we show that the known function $\mathcal{G}^\omega(n)$ can be written as the annealed, deterministic centring with an environment dependent correction where this correction is a sum of centred i.i.d. random variables (with non-zero variance) under the environment law. This shows that the correction obeys a central limit theorem under $\mathbf{P}$ which means that it has $\sqrt{n}$ fluctuations and is, therefore, necessary.

A quenched central limit theorem has not been proved for the randomly trapped random walk in $\mathbb{Z}^d$ for $d \geq 2$ although it has been shown in [27] that, for the unbiased walk, if $\mathbb{E}[\eta_0] < \infty$ then $X_{nt}n^{-1/2}$ converges in distribution to a scaled Brownian motion under the annealed law. Using the technique developed in [21] it should be possible to extend this to a quenched result in dimension $d \geq 4$ without the need for an environment dependent centring; similarly, this should also hold for the unbiased case. In dimensions $d \leq 3$ the technique cannot be applied because it relies on two independent copies of the walk not visiting the same vertices at large times. By comparison with the Bouchaud trap model (see [7]) we expect that a quenched functional central limit theorem should still hold (at least in the unbiased case) for $d = 2, 3$.

In Chapter 5 we consider the randomly trapped random walk when the trapping is too strong to obtain a central limit theorem. In this setting there are two distinct phases depending on whether $\mathbb{E}[\eta_0] < \infty$. In both cases we prove functional limiting results for the position of the walk.

When $\mathbb{E}[\eta_0] = \infty$ we have that the walk is sub-ballistic. In this setting, our main result is Theorem 5.1 in which we show that if $\beta > 1$, there exists a_n regularly varying with index $1/\alpha$ for some $\alpha \in (0,1)$ and for any $k \in \mathbb{N}$ and some function f we have that

$$-n\log\left(\mathbf{E}\left[E^\omega\left[\exp\left(\frac{\lambda\eta_0}{a_n}\right)\right]^k\right]\right) \sim f(k)\lambda^\alpha$$

then $X_{a_n t}/n$ converges in $\mathbb{P}$-distribution to the inverse of an α-stable subordinator.

The other phase we consider is where $\mathbb{E}[\eta_0] < \infty$ but $\mathbb{E}[\eta_0^2] = \infty$. In this setting we are able to apply the speed result from Chapter 3 but the fluctuations are too large to obtain a central limit theorem. The aim of this case is to prove Theorem 5.2. Here, we show that if $\beta > 1$, there exists a_n regularly varying with index $1/\alpha$ for some $\alpha \in (1, 2)$ and for any $k \in \mathbb{N}$ and some function f we have that

$$n \log \left(\mathbf{E} \left[E^\omega \left[\exp \left(\frac{\lambda(\eta_0 - \mathbb{E}[\eta_0])}{a_n} \right) \right]^k \right] \right) \sim f(k)\lambda^\alpha$$

then $(X_{nt} - nt\nu_\beta)/a_n$ converges in $\mathbb{P}$-distribution to a stable process with index α.

These two asymptotic conditions for the sub-diffusive regimes appear to be quite technical however they relate to the usual stable law conditions (as we discuss in Chapter 5) and are, in fact, quite applicable as we shall show in a range of examples. Importantly, these conditions only depend on a single excursion. This makes dealing with the quenched law much more straightforward since we no longer need to understand the correlation between the number of times traps are entered.

1.2 Random walks on trees

Our main motivation for investigating the randomly trapped random walk model is the study of biased random walks on GW-trees conditioned to survive. A GW-tree conditioned to survive consists of an infinite backbone with finite trees attached as branches. These branches form dead-ends in the environment which makes it a natural setting for observing trapping as the walk is slowed by taking excursions in the branches of the tree.

In this section we introduce the general framework of random walks on trees with particular focus on GW-trees. We then briefly describe the main results that we later prove for these models. By a tree we mean a rooted, locally finite graph $\mathscr{T}$ which contains no cycles and we let d be the graph distance metric on $\mathscr{T}$. We denote by ρ the root and refer to the collection of vertices of graph distance k from ρ as the k^{th} generation of the tree. For vertices $x, y \in \mathscr{T}$ which are neighbours (i.e. $d(x, y) = 1$) we say that x is the parent of y (equivalently y is a child of x) if $d(\rho, x) = d(\rho, y) - 1$; that is, x belongs to the generation before y. We then let $c(x)$ be the set of children of x and $\overleftarrow{y}$ to be the unique parent of y (when $y \neq \rho$). We say that y is a descendant of x (equivalently x is an ancestor of y) if the unique self-avoiding path between ρ and y passes through x. We then denote by $\mathscr{T}_x$ the maximal subtree formed by the descendants of x (where we consider $\mathscr{T}_x$ to be rooted at x). We define $\mathcal{H}(\mathscr{T}) := \sup\{d(\rho, x) : x \in \mathscr{T}\}$ to be the height of the tree $\mathscr{T}$.

We now introduce the biased random walk on a fixed tree $\mathscr{T}$. A β-biased random walk on $\mathscr{T}$ is a random walk $(X_n)_{n\geq 0}$ on the vertices of $\mathscr{T}$ which is β-times more likely to make a transition to a given child of the current vertex than the parent (which are the only options). More specifically, the random walk started from $z \in \mathscr{T}$ is the Markov chain defined by $P_z^{\mathscr{T}}(X_0 = z) = 1$ and the transition probabilities

$$P_z^{\mathscr{T}}(X_{n+1} = y | X_n = x) = \begin{cases} \frac{1}{1+\beta|c(x)|}, & \text{if } y = \overleftarrow{x}, \\ \frac{\beta}{1+\beta|c(x)|}, & \text{if } y \in c(x),\ x \neq \rho, \\ \frac{1}{|c(\rho)|}, & \text{if } y \in c(x),\ x = \rho, \\ 0, & \text{otherwise.} \end{cases}$$

We use $\mathbb{P}(\cdot) := \int P_\rho^{\mathscr{T}}(\cdot)\mathbf{P}(\mathrm{d}\mathscr{T})$ for the annealed law obtained by averaging the quenched law $P_\rho^{\mathscr{T}}$ over a law $\mathbf{P}$ on random rooted trees. Unless indicated otherwise, we start the walk at ρ.

For $x \in \mathscr{T}$, let $|x| := d(\rho, x)$ denote the distance between x and the root of the tree. Our principle interest will be the evolution of $|X_n|$ with respect to both $\mathbb{P}$ and $P^{\mathscr{T}}$.

We next briefly describe GW-processes and the GW-trees which arise from them; for further detail see, for example, [44]. Let $\{p_k\}$ denote a probability distribution on $\mathbb{Z}^+$, $f(s) := \sum_{k\geq 0} p_k s^k$ its probability generating function and ξ a random variable with this law. To avoid trivial cases we assume that $p_0 + p_1 < 1$. We consider a GW-process with offspring distribution ξ as the Markov chain $(Z_n)_{n\geq 0}$ describing the generation sizes of a branching process started from a single progenitor ($Z_0 = 1$) where each individual independently gives rise to a random number (distributed with respect to f) of offspring in the next generation. That is,

$$Z_{n+1} = \sum_{j=1}^{Z_n} \xi_{n,j}$$

where $\{\xi_{n,j} : n \geq 0, j \geq 1\}$ are independent copies of ξ. Let $\mu := \mathbf{E}[\xi]$ be the mean of the offspring distribution which we assume to be finite and $\sigma^2 := \mathrm{Var}_{\mathbf{P}}(\xi)$ which may be infinite. (In fact, so that the asymptotic (2.9) holds, we make the slightly stronger assumption throughout that $\mathbf{E}[\xi \log^+(\xi)] < \infty$.) This process gives rise to a random tree $\mathcal{T}^f$ where individuals are represented by vertices, edges connect individuals with their offspring and we identify the unique progenitor with the root ρ.

We write q for the extinction probability of Z_n; that is, the probability that $Z_n = 0$ eventually. It is classical (e.g. [4, Section I.5]) that $q < 1$ if and only if $\mu > 1$; that is, if $\mu \leq 1$ then the process dies out $\mathbf{P}$-a.s. and, otherwise, there is some positive probability that the process survives forever. We refer to the three cases $\mu < 1$, $\mu = 1$ and $\mu > 1$ as the subcritical, critical and supercritical cases respectively.

1.2.1 Random walks on supercritical Galton-Watson trees

As previously mentioned, a supercritical GW-tree comes from a GW-process whose offspring distribution has mean $\mu > 1$ and, therefore, there is a positive probability that the tree survives for infinitely many generations. We will work on these trees and denote by $\mathcal{T}$ a random tree with the law of $\mathcal{T}^f$ conditioned on survival.

We next briefly describe the supercritical GW-tree conditioned on survival following [4, Section I.12] and [44]. We define

$$g(s) := \frac{f(s) - f(qs)}{1 - q} \qquad \text{and} \qquad h(s) := \frac{f(qs)}{q}$$

which are generating functions of GW-processes. In particular, g is the generating function of a GW-process without deaths and h is the generating function of a subcritical GW-process. An f-GW-tree conditioned on nonextinction $\mathcal{T}$ can be generated by first generating a g-GW-tree $\mathcal{T}^g$ and then, to each vertex x of $\mathcal{T}^g$, appending a random number $\mathcal{M}_x$ of independent h-GW-trees. We refer to $\mathcal{Y} := \mathcal{T}^g$ as the backbone of $\mathcal{T}$, the finite trees appended to $\mathcal{Y}$ as the traps and the vertices in the first generation of the traps as the buds. The distribution of $\mathcal{M}_x$ depends on the backbone locally; we will not use the exact form but we include it here for brevity. Specifically, let $c_g(x)$ be the offspring of x in $\mathcal{T}^g$ then the distribution of $\mathcal{M}_x$ conditional on $\mathcal{T}^g$ can be defined by its probability generating function:

$$\mathbf{E}\left[s^{\mathcal{M}_x} | \mathcal{T}^g\right] = \frac{f^{(|c_g(x)|)}(qs)}{f^{(|c_g(x)|)}(q)}$$

where $f^{(k)}$ denotes the k^{th} derivative of f.

It has been shown in [55] that if $\beta \in (\mu^{-1}, f'(q)^{-1})$ then $|X_n|n^{-1}$ converges $\mathbb{P}$-a.s. to a deterministic constant $\nu_\beta > 0$ called the speed of the walk. We refer to this as the ballistic regime. When $\beta < \mu^{-1}$ the walk is recurrent and $|X_n|n^{-1}$ converges $\mathbb{P}$-a.s. to 0. This occurs because the backbone has average degree μ and, therefore, the embedded walk on the backbone only has drift away from the root when $\beta > \mu^{-1}$. When the bias is large the walk is transient but slowed by having to make long sequences of movements against the bias in order to escape the traps; in particular, if $\beta \geq f'(q)^{-1}$ then the slowing effect is strong enough to cause $|X_n|n^{-1}$ to converge $\mathbb{P}$-a.s. to 0. We refer to this as the sub-ballistic regime.

In the ballistic regime $|X_n|n^{-1}$ converges $\mathbb{P}$-a.s. to a deterministic constant $\nu_\beta > 0$; in this case it is natural to study the fluctuations. For $\varsigma, t > 0$ and $n = 1, 2, ...$ define

$$B_t^n := \frac{|X_{\lfloor nt \rfloor}| - nt\nu_\beta}{\varsigma\sqrt{n}}.$$

It has been shown in [65] that if the offspring law has no deaths ($p_0 = 0$) and exponential moments then for any $\beta > \mu^{-1}$ there exists $\varsigma > 0$ such that, for $\mathbf{P}$-a.e. tree $\mathcal{T}$, the process $(B^n_t)_{t\geq 0}$ converges in $P^{\mathcal{T}}$-distribution to a standard Brownian motion. This no longer requires the ballisticity condition $\beta < f'(q)^{-1}$ which is due to the trapping in the dead-ends caused by the leaves. Indeed, when $p_0 = 0$ we have that $q = 0$ and $f'(0) = 0$ therefore the condition becomes irrelevant. This regime is studied further to the case where $\beta = \mu^{-1}$ and $p_0 = 0$; in this setting $\nu_\beta = 0$ and it is shown that $(B^n_t)_{t\geq 0}$ converges in $P^{\mathcal{T}}$-distribution to the absolute value of a Brownian motion.

This result is extended in [31] to random walks on multi-type GW-trees with leaves. That is, the offspring distribution at each vertex is chosen randomly (depending on the past) from some finite alphabet and the bias is fixed as the inverse of the Perron-Frobenius eigenvalue for the matrix of expected offspring numbers. In this way the walk has zero drift along the backbone $\mathbf{P}$-a.s. Although the dead-ends in this model trap the walk, the bias is small and therefore the slowing is weak.

It has been hypothesized in [9, Conjecture 3.1] that if the offspring law has deaths ($p_0 > 0$) then, for any $\beta \in (\mu^{-1}, f'(q)^{-1/2})$ and $\mathbf{P}$-a.e. tree $\mathcal{T}$, the process $(B^n_t)_{t\geq 0}$ should converge in $P^{\mathcal{T}}$-distribution to a standard Brownian motion. By choosing $p_0 > 0$, we allow the tree to have leaves; this creates traps in the environment which slow the walk. We then require this additional upper bound on the bias so that the trapping times have finite variance. We prove this conjecture in Section 6.1 and conclude that this upper bound on the bias is sharp.

In the sub-ballistic phase, $\beta > f'(q)^{-1}$, it has further been observed in [10] that if the offspring distribution has finite variance, then the walker follows a polynomial escape regime. In Section 6.2 we show that this regime extends to the case where the offspring distribution belongs to the domain of attraction of a stable law with index $\alpha \in (1, 2)$.

We shall see that the finite branches are typically quite short and the embedded walk on the backbone does not deviate too far from the furthest point reached; we therefore have a strong relation between $|X_n|$ and the time $\Delta_n := \inf\{m \geq 0 : X_m \in \mathcal{Y}, |X_m| = n\}$ taken to reach the n^{th} generation of the backbone. The time Δ_n mainly consists of the duration of excursions in the deepest branches. Due to the transience of the walk we have that the amount of time spent in these deep branches are asymptotically independent. It follows that Δ_n can be approximated by the sum of i.i.d. copies of the time spent in independent large branches.

We show further that only the height $\mathcal{H}$ of the branch, and not the foliage, contributes to the scaling. By comparison with the model in which we strip all of the branch except the unique self-avoiding path to the deepest point, by transience the walk reaches the deepest point with positive probability and then takes a geometric number of short excursions with escape probability close to $\beta^{-\mathcal{H}}$. In particular, this means that the time spent in a branch of height $\mathcal{H}$ will cluster around $\beta^{\mathcal{H}}$.

The height $\mathcal{H}$ is approximately geometric with parameter $f'(q)$; we therefore see branches of height approximately $\log(n)/\log(f'(q)^{-1})$ by generation n. In particular, the time spent in the largest branch up to generation n will cluster around $\beta^{\log(n)/\log(f'(q)^{-1})} \approx n^{1/\gamma}$ where we define the exponent

$$\gamma := \frac{\log(f'(q)^{-1})}{\log(\beta)}. \tag{1.1}$$

We then have that $\Delta_n n^{-1/\gamma}$ converges in distribution, with respect to $\mathbb{P}$, along subsequences of the form $n_l(t) = \lfloor tf'(q)^{-l} \rfloor$. This shows that $|X_n|$ scales with n^{γ}.

It is natural to consider whether this result can be extended to show that $|X_n|n^{-\gamma}$ converges more generally. It has been shown in [10] that this is not the case due to a certain lattice effect. That is, since $\mathcal{H}$ is approximately geometric we have that $\beta^{\mathcal{H}}$ will not belong to the domain of attraction of any stable law therefore Δ_n only converges along specific subsequences. A related model is studied in [11] and [43] where the conductance along each edge is chosen randomly according to a distribution satisfying a certain non-lattice assumption. In this setting the tail of the trapping times obey a pure power law and the rescaled walk converges in distribution.

We will not study the regime where $\beta \in (f'(q)^{-1/2}, f'(q)^{-1})$ here. In this regime the walk has a positive speed ν_β but the slowing is too strong to obtain a central limit theorem. Heuristically, we expect that $|X_n - n\nu_\beta| \simeq n^{1/\gamma}$ where γ is the exponent defined in (1.1). However, due to the same lattice effect seen in the sub-ballistic regime, we would only expect to observe convergence along suitably chosen subsequences.

1.2.2 Random walks on subcritical Galton-Watson trees

Similarly to the supercritical GW-tree conditioned to survive, a subcritical GW-tree conditioned to survive consists of an infinite backbone with finite trees attached as branches. These branches are formed of collections of subcritical GW-trees as in the supercritical case however, the backbone consists of a semi-infinite path emanating from a fixed root vertex. In particular, the branches of the subcritical tree are i.i.d. and very similar in structure to those of the supercritical tree. For this reason, studying random walks on the subcritical tree is a natural tool for studying the trapping mechanism of random walks on the supercritical tree without the added complications which arise from the backbone of the supercritical tree. The study of random walks on subcritical trees is largely motivated by the study of random walks on supercritical trees; despite this, we will see that the subcritical tree exhibits various unusual characteristics which are not seen on the supercritical tree and give rise to interesting asymptotic properties for the walk.

The branches of the subcritical trees are typically very short and the embedded walk on the backbone does not spend much time close to the root; using this, we show in Section 2.5 that a random walk on a subcritical GW-tree conditioned to survive can be coupled to a randomly trapped random walk so that the two walks do not deviate too far from each other. We therefore use random walks on subcritical GW-trees conditioned to survive as our main motivation for studying one-dimensional randomly trapped random walks.

We now describe subcritical GW-trees conditioned to survive following [2], [46] and [49]. Unlike the supercritical GW-tree, the unconditioned subcritical GW-tree dies out $\mathbf{P}$-a.s. and therefore, *a priori*, we cannot discuss subcritical GW-trees conditioned to survive without first questioning their existence. It has been shown in [49] that there is a well defined probability measure over f-GW trees conditioned to survive for infinitely many generations which arises as a limit of probability measures over f-GW trees conditioned to survive at least n generations. It is this distribution we consider and we denote by $\mathcal{T}$ a tree with this law.

Recall that the offspring law of the process is given by $\mathbf{P}(\xi = k) = p_k$, we then define the size-biased distribution by the probabilities $\mathbf{P}(\xi^* = k) = kp_k\mu^{-1}$. It can be seen (e.g. [46]) that the subcritical GW-tree conditioned to survive coincides with a construction of a random tree with two types of vertex which we refer to as *normal* and *special* vertices. Start with a single special vertex in generation 0. At each generation let every normal vertex give birth onto vertices according to independent copies of the offspring distribution and (independently) every special vertex give birth onto vertices according to independent copies of the size-biased distribution. We then choose one child of each special vertex uniformly at random to be special. All remaining vertices (children of either normal or special vertices) are then labelled as normal. We must have a unique special vertex in each generation because we start with a single special vertex and each special vertex gives birth to precisely one special vertex.

Unlike the supercritical tree which has infinitely many infinite paths, the backbone $\mathcal{Y}$ of the subcritical tree conditioned to survive consists of a unique semi-infinite path from the initial vertex ρ. Specifically, $\mathcal{Y}$ is formed of the special vertices in the construction. As for the supercritical tree, we refer to those vertices not on $\mathcal{Y}$ which are children of vertices on $\mathcal{Y}$ as buds and the finite trees rooted at the buds as traps. We write ρ_i for the unique vertex on $\mathcal{Y}$ which is distance i from ρ and $\mathcal{T}^{*-}_{\rho_i} := \mathcal{T}_{\rho_i} \setminus \mathcal{T}_{\rho_{i+1}}$ as the branch emanating from ρ_i. Due to the one dimensional backbone, this model easily fits into the randomly trapped random walk framework and as such it will be convenient to consider the walk as a trapping model. To this end, we define the embedded walk $(Y_k)_{k\geq 0}$ given by $Y_k := X_{S_k}$ where $S_0 := 0$ and $S_k := \inf\{m > S_{k-1} : \ X_m, X_{m-1} \in \mathcal{Y}\}$ for $k \geq 1$. This means that Y makes the same transitions along the backbone as X but does not experience the traps.

Before describing our results for random walks on subcritical GW-trees conditioned to survive let us briefly discuss how these trees differ from the supercritical

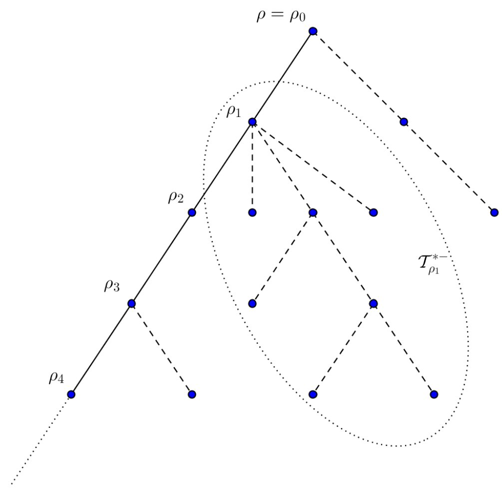

Figure 1.1: A sample subcritical GW-tree conditioned to survive $\mathcal{T}$ with the backbone $\mathcal{Y}$ represented by solid lines and the buds and traps connected by dashed lines.

GW-trees conditioned to survive. The most obvious difference is the structure of the backbone. In the subcritical case we have a semi-infinite path emanating from a fixed root and in the supercritical case we have a random tree. This means that dealing with the embedded walk in the subcritical case is far easier however it will also give rise to interesting phenomena when we study quenched central limit theorems in Chapter 3.

Another key difference is the distribution over the buds. In the subcritical case, the number of buds had a size-biased law independent of the position on the backbone. In the supercritical case, the distribution over the number of buds is more complicated since it depends on the backbone. Importantly, in the supercritical case, the expected number of buds can be bounded above by $\mu(1-q)^{-1}$ independently of higher moments of the offspring law. This is not true in the subcritical case. In particular, if ξ has infinite second moments then ξ^* has infinite mean. This will be important in Chapter 4 where we will observe rich behaviour in the subcritical case (when the offspring law has finite mean but infinite variance) which is not present in the supercritical case.

A final noteworthy difference is the offspring distribution of the traps. Recall that in both cases the buds are roots of subcritical GW-trees. In the subcritical case the law over these trees is the same as the original offspring law. In the supercritical case the law coincides with that of the supercritical GW-process conditioned to die out; in particular, it is governed by h. Letting ξ^h denote a variable with this law we have that $\mathbf{P}(\xi^h = k) = p_k q^{k-1}$ and therefore ξ^h has exponential moments.

When the offspring law has finite variance, the limiting behaviour of $|X_n|$ on the supercritical and subcritical trees conditioned to survive is very similar. Both have a regime with linear scaling and a regime with polynomial scaling caused by the same phenomenon of deep traps. When the offspring law has infinite variance, the bud distribution of the subcritical tree has infinite mean which causes an extra slowing effect which is not seen with the supercritical tree. This equates for the different exponents observed in the two models as shown in Figure 1.2. The walk on the critical tree experiences a similar trapping mechanism to the subcritical tree; however, the slowing is more extreme and belongs to a different universality class which had been shown in [28] to yield a logarithmic escape rate.

The first order scaling limits that can occur in the subcritical case are as follows. There exists a limiting speed ν_β such that $|X_n|/n$ converges $\mathbb{P}$-a.s. to ν_β under $\mathbb{P}$; moreover, the walk is ballistic if and only if $1 < \beta < \mu^{-1}$ and $\sigma^2 < \infty$. We prove this in Chapter 3 by comparison with the randomly trapped random walk. In fact, we are able to prove the following explicit expression for the speed:

$$\nu_\beta = \frac{\mu(\beta-1)(1-\beta\mu)}{\mu(\beta+1)(1-\beta\mu)+2\beta(\sigma^2-\mu(1-\mu))}.$$

Such an expression is not known in the supercritical case; however, a description of the invariant distribution of the environment seen from the particle is used in [1] to give an expression of the speed in terms of an annealed expectation.

The sub-ballistic regime has four distinct phases. When $\beta \leq 1$ the walk is recurrent and we are not concerned with this case here; we study the remaining three cases in Chapter 4. Figure 1.2 is the phase diagram for the almost sure limit of $\log(|X_n|)/\log(n)$ (which is the leading order polynomial exponent in the scaling of $|X_n|$ relative to β and μ) where the offspring law has stability index α (which is 2 when $\sigma^2 < \infty$). Strictly, $f'(q)$ is not a function of μ therefore the line $\beta = f'(q)^{-1}$ is not well defined; Figure 1.2 shows the particular case when the offspring distribution belongs to the geometric family. It is always the case that $f'(q) < 1$ therefore some such ballistic region always exists however the parametrisation depends on the family of distributions.

As in the supercritical case, X_n has a strong relationship with the time taken to reach the n^{th} level of the backbone $\Delta_n := \inf\{m \geq 0 : X_m \in \mathcal{Y}, |X_m| = n\}$ and we consider this for much of the work in sub-ballistic regimes. When $1 < \beta < \mu^{-1}$ and $\sigma^2 = \infty$ the expected time spent in a trap is finite and the slowing of the walk is due to the large number of buds. That is, the bud distribution has infinite mean which results in the walk making a large number of short excursions into the branches. We show that if the offspring law belongs to the domain of attraction of a stable law with index $\alpha \in (1, 2)$ then Δ_{nt} can be scaled so that it converges in distribution to an $\alpha - 1$ stable subordinator. This implies that $|X_n| \simeq n^{\alpha-1}$.

When $\beta\mu > 1$ and $\sigma^2 < \infty$, the expected time spent in a subcritical GW-tree forming a trap is infinite because the strong bias forces the walk deep into traps and long sequences of movements against the bias are required to escape. This is the same phenomena that occurs in the supercritical case and we can extend our definition of the exponent γ from (1.1) to

$$\gamma := \begin{cases} \frac{\log(f'(q)^{-1})}{\log(\beta)}, & \mu > 1, \\ \frac{\log(\mu^{-1})}{\log(\beta)}, & \mu < 1, \end{cases} \tag{1.2}$$

to see that $\Delta_{nt} n^{-1/\gamma}$ converges in distribution along certain subsequences. This implies that $|X_n| \simeq n^\gamma$. It is noteworthy here that the $f'(q)$ occurring in the exponent for the supercritical case is replaced by μ in the subcritical case. These constants are the mean number of offspring from vertices in traps of the supercritical and subcritical trees respectively. This is important because it determines the height of the branch which is fundamental to the trapping.

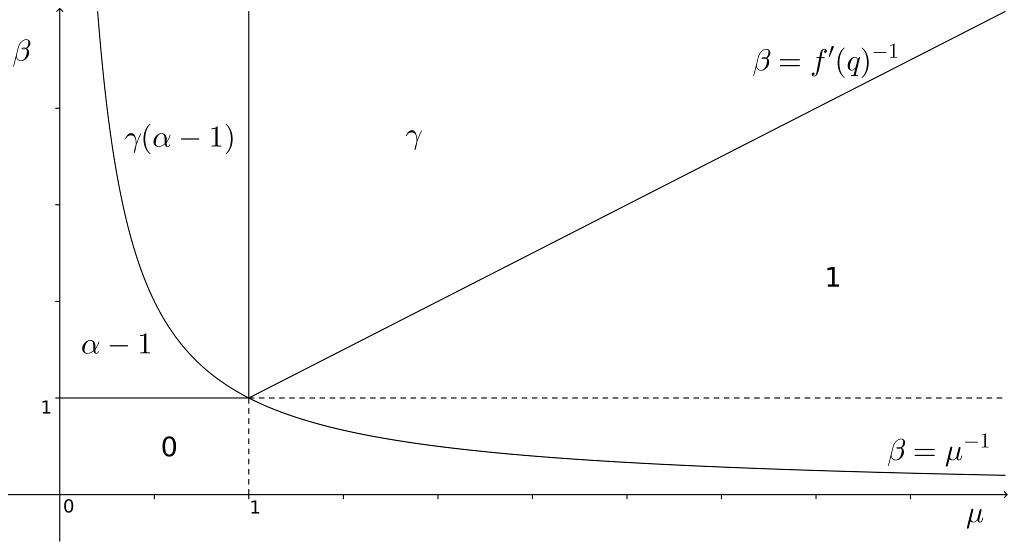

Figure 1.2: Phase diagram for the leading order polynomial exponent in the scaling of the walk relative to the mean of the offspring law and bias of the walk.

In the final case for the subcritical tree ($\beta\mu > 1$, $\sigma^2 = \infty$) slowing effects are caused by both strong bias and the large number of buds. Naïvely, one may expect that only the stronger of the two slowing effects take place. However, this is not the case. Both effects are caused by structural properties of the largest branches; that is, the breadth and the height. These two properties are strongly related; if a GW-tree has a large number of vertices in its first generation then it is likely to survive for many generations. In this regime we observe similar trapping to the case when $\beta\mu > 1$ and $\sigma^2 < \infty$. The major change is that the trees are significantly taller due to the different bud distribution. In particular, we show that Δ_{nt} can be rescaled so that it converges in distribution along subsequences. This gives us that $|X_n| \simeq n^{\gamma(\alpha-1)}$.

In Chapter 3 we use results for the randomly trapped random walk to deduce results for random walks on subcritical GW-trees conditioned to survive. In particular, further to the speed result mentioned above, we show an annealed functional central limit theorem and a quenched central limit theorem with an environment dependent centring. The annealed result emulates the corresponding result for the random walk on the supercritical GW-tree conditioned to survive. That is, under an additional moment condition on the offspring distribution, we show that for some $\varsigma > 0$

$$B_t^n := \frac{|X_{nt}| - nt\nu_\beta}{\varsigma\sqrt{n}}$$

converges in $\mathbb{P}$-distribution to a standard Brownian motion when $\beta \in (1, \mu^{-1/2})$. As in the supercritical case, we require an additional upper bound on the bias to deal with the trapping in finite trees which we show to be sharp.

The quenched outcome is considerably different to the corresponding result in the supercritical case. The result is similar to that of the randomly trapped random walk. We prove that, under an additional moment condition on the offspring distribution, for some $\vartheta > 0$ and $\mathbf{P}$-a.e. $\mathcal{T}$ we have that

$$\frac{|X_n| - \mathcal{G}^{\mathcal{T}}(n)}{\vartheta\sqrt{n}}$$

converges in $P^{\mathcal{T}}$-distribution to a standard Gaussian when $\beta \in (1, \mu^{-1/2})$ where $\mathcal{G}^{\mathcal{T}}$ is an environment dependent centring. This centring $\mathcal{G}^{\mathcal{T}}(n)$ is equal to the centring in the annealed case ($nt\nu_\beta$) added to a sum of n variables which are fixed under the quenched law but i.i.d. with non-zero variance under the environment law. From this, it is clear that this sum obeys a central limit theorem and, therefore, this environment dependent centring is necessary for the quenched result. In the supercritical case we observed convergence to a Brownian motion (similar to the annealed case) without the need of the environment dependent centring. The reason for this disparity is due to the one-dimensional backbone. In the subcritical case, the walk P-a.s. visits the root of every trap in the tree. This results in the walk perceiving the specific environment on the fluctuation level. In the supercritical case the walk will randomly choose one of infinitely many escape routes; this creates a mixing of the environment which yields a deterministic centring in the quenched result.

Figure 1.3 is the phase diagram indicating the regimes in which the walk can obey a central limit theorem. Whether or not the walk does obey a central limit theorem depends on the moments of the offspring distribution; however; our results show that if the offspring law has finite third moments or exponential moments in the subcritical and supercritical cases respectively then the walk obeys an annealed functional central limit theorem for the entirety of the region.

We do not study the regime where $\beta \in (\mu^{-1/2}, \mu^{-1})$ directly; however, we

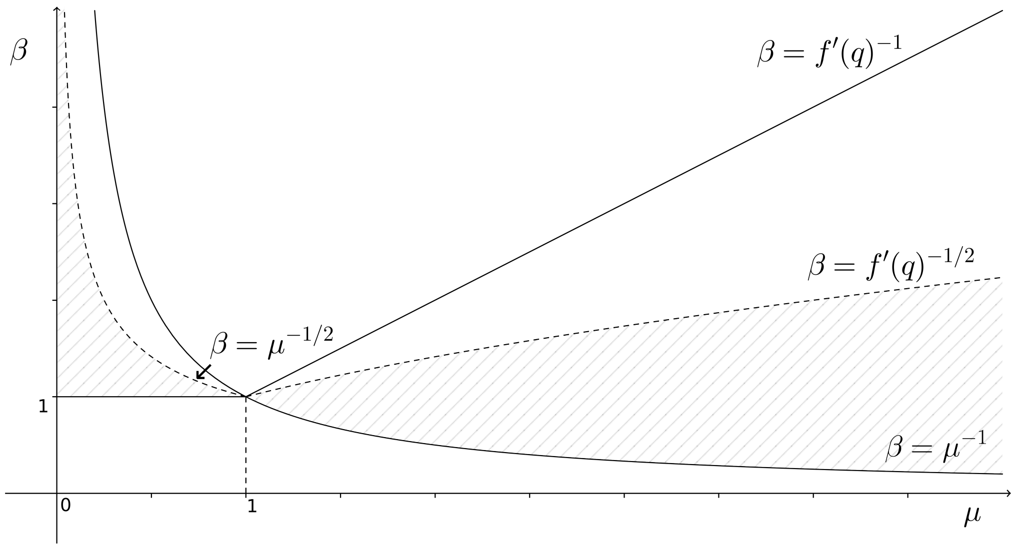

Figure 1.3: Phase diagram with the shaded region indicating the combinations of μ and β such that the walk obeys a central limit theorem.

consider a related process in Chapter 5. In this chapter we study the randomly trapped random walk when the holding times have finite mean and infinite variance and show conditions under which the centred and rescaled walk converges to a stable process. We apply this result to the comb model which can be seen as the subcritical GW-tree model where each branch is pruned so that it consists only of the unique path to the deepest vertex. As in the sub-ballistic regime, we expect that this structure should determine the correct scaling of the walk on the subcritical tree; in particular, it suggests that $|X_n - n\nu_\beta| \simeq n^{1/\gamma}$. Similarly to the sub-ballistic regime for the subcritical GW-tree, we only observe convergence along certain subsequences in the comb model due to the lattice effect.

Chapter 2

Preliminaries

In this chapter we provide the necessary background and several technical results that we use throughout the thesis. We do not prove any new results in Section 2.1 but give a brief overview of Skorohod topologies which will play an important role in much of the thesis. In Section 2.2 we state some well known formulas concerning stable laws and prove a straightforward technical lemma. We cover basic estimates relating to random walks in Section 2.3. This includes the classical Gambler's ruin, expressions for expected cover times and an upper bound on the correlation between the number of visits to two vertices in a tree. In Section 2.4 we address branching processes by stating some classical formulas and proving two technical results which we use throughout. We conclude the chapter in Section 2.5 by proving that we can couple a random walk on a subcritical GW-tree conditioned to survive with a randomly trapped random walk so that the two walks do not deviate too far from each other.

2.1 Skorohod topologies

In this section we briefly outline the space of càdlàg functions, the Skorohod J_1, M_1 and M_2 topologies and several useful results that we apply throughout the thesis. Everything stated here can be found in [76] to which we direct the reader for more detail. We also refer the reader to [18] and [45] which give a good account of convergence of stochastic processes in this setting.

In short, the space of continuous functions is not a suitable choice to describe many of the processes we consider which must contain jumps. We, therefore, consider instead the space of càdlàg functions $D([0,T],\mathbb{R})$ mapping $[0,T]$ onto $\mathbb{R}$; that is, those functions which are right continuous and have left limits.

Because we consider processes with discontinuous paths, the uniform topology will not always be an appropriate choice. We consider, instead, the Skorohod topologies which consider functions to be close if they can be mapped onto one another by a uniformly small perturbation in time and space as opposed to solely uniformly small perturbations in space reflected by the uniform topology.

Let Λ be the set of strictly increasing continuous functions mapping $[0,T]$ onto itself. We then consider the Skorohod J_1 metric d_{J_1} on $D([0,T],\mathbb{R})$ defined by

$$d_{J_1}(f,g) := \inf_{\lambda\in\Lambda} \sup_{t\in[0,T]} \left(|f(t)-g(\lambda(t))| + |t-\lambda(t)|\right). \tag{2.1}$$

For $f \in D([0,T],\mathbb{R})$ let $\Gamma_f := \{(z,t) \in \mathbb{R}\times[0,T] : z = \alpha f(t^-) + (1-\alpha)f(t) \text{ for some } \alpha \in [0,1]\}$ be the completed graph of f. We then define an ordering on Γ_f by saying that $(z_1,t_1) \leq (z_2,t_2)$ if either $t_1 < t_2$ or $t_1 = t_2$ and $|f(t_1^-) - z_1| \leq |f(t_2^-) - z_2|$. A parametric representation of Γ_f is a non-decreasing function $u = (u_1,u_2) : [0,T] \to \Gamma_f$. Let Π_f denote the parametric representations of Γ_f then we define the Skorohod M_1 metric d_{M_1} on $D([0,T],\mathbb{R})$ as

$$d_{M_1}(f,g) := \inf_{u\in\Pi_f, v\in\Pi_g} \left(|u_1-v_1| \vee |u_2-v_2|\right).$$

This is weaker than the J_1 topology; in particular, it allows a discontinuous jump and a continuous surge to be close which the J_1 topology does not.

Finally, we define the Skorohod M_2 distance by

$$\begin{aligned} d_{M_2}(f,g) := &\sup_{(z_f,t_f)\in\Gamma_f} \inf_{(z_g,t_g)\in\Gamma_g} \left(|z_f-z_g| \vee |t_f-t_g|\right) \\ &\vee \sup_{(z_g,t_g)\in\Gamma_g} \inf_{(z_f,t_f)\in\Gamma_f} \left(|z_f-z_g| \vee |t_f-t_g|\right). \end{aligned}$$

This is weaker than the M_1 topology since it only requires that all points on the completed graph of f are close to some other point on the completed graph of g.

We extend this notion to $D := D([0,\infty),\mathbb{R})$ by characterising convergence in the respective topologies. We say that $f_n \to f$ in $D([0,\infty),\mathbb{R})$ with the Skorohod J_1 (or M_1, M_2) topology if and only if $f_n \to f$ in in $D([0,T],\mathbb{R})$ with the Skorohod J_1 (or M_1, M_2) topology for every continuity point T of f. Furthermore, for convenience, we will often write $D_{J_1}([0,\infty),\mathbb{R})$ to denote $D([0,\infty),\mathbb{R})$ equipped with the Skorohod J_1 topology and likewise with U, M_1 or M_2.

We now state [76, Theorems 13.2.1, 13.2.2, 13.6.3, 13.7.1 & Corollary 13.6.4] which will be used throughout the thesis. We write $D_\uparrow, D_{\uparrow\uparrow}$ to denote the subsets of D which are increasing and strictly increasing respectively. We then write $D_u, D_{u,\uparrow}, D_{u\uparrow\uparrow}$ for those subsets of unbounded functions. Furthermore, $C, C_\uparrow, C_{\uparrow\uparrow}$ denote the corresponding subsets of continuous functions.

Proposition 2.1.1. *Let (D,J_1) denote D equipped with the J_1 topology and similarly with J_1 replaced by one of M_1, M_2 or U where U denotes the uniform topology.*

i) (continuity of composition at continuous limits) The composition map from $D\times D_\uparrow$

to D is measurable and continuous at $(f,g) \in C \times C_{\uparrow}$;

ii) (J_1-continuity of composition) The composition map from $D \times D_{\uparrow}$ to D taking (f,g) into $(f \circ g)$ is continuous at $(f,g) \in (C \times D_{\uparrow}) \cup (D \times C_{\uparrow\uparrow})$ using the J_1 topology throughout;

iii) (equivalent characterisations of convergence for monotone functions) Suppose that $(f_n)_{n\geq 1}$, $f \in D_{u,\uparrow}$, then the following are equivalent:

(a) $f_n \to f$ in $D_{u\uparrow}$ with the M_1 topology;

(b) $f_n \to f$ for all t in a dense subset of $(0,\infty)$;

(c) $f_n^{-1} \to f^{-1}$ in $D_{u\uparrow}$ with the M_1 topology;

(d) $f_n^{-1} \to f^{-1}$ for all t in a dense subset of $(0,\infty)$;

iv) (inverse with linear centring) Suppose $c_n(f_n - e) \to f$ as $n \to \infty$ in $D([0,\infty),\mathbb{R})$ with one of the topologies M_2, M_1 or J_1 where $f_n \in D_u$, $c_n \to \infty$, $f(0) = 0$ and e denotes the identity map.

(a) If the topology is M_2 or M_1, then $c_n(f_n^{-1} - e) \to -f$ as $n \to \infty$ with the same topology.

(b) If the topology is J_1 and f has no positive jumps, then $c_n(f_n^{-1} - e) \to -f$ as $n \to \infty$ with the J_1 topology.

v) (continuity of the inverse at strictly increasing functions) The inverse map from (D_u, M_2) to $(D_{u\uparrow}, U)$ is measurable and continuous at $f \in D_{u\uparrow\uparrow}$.

2.2 Stable laws

Throughout the thesis we will use a range of properties about stable laws and their domains of attraction. We state them here along with several new results that will be useful later. For a more detailed introduction and proofs of some of these facts, we direct the reader to [15] and [34].

We begin with a note concerning slowly and regularly varying functions. We say that a function L varies slowly (at ∞) if for any $c > 0$ we have that

$$\lim_{x\to\infty} \frac{L(cx)}{L(x)} = 1$$

and that the function R varies regularly (with index α) if there exists a slowly varying function L such that $R(x) = x^{\alpha} L(x)$.

For the purposes of this section let $\xi, \{\xi_k\}_{k\geq 1}$ be i.i.d. with law F and define $S_n := \sum_{k=1}^{n} \xi_k$. We say that F is stable if for each $n \in \mathbb{N}$ there exist constants $a_n > 0$,

b_n such that $S_n \stackrel{\mathrm{d}}{=} a_n\xi + b_n$ and F is not concentrated at a single point. By [34, Theorem VI.1.1] we have that $a_n = n^{1/\alpha}$ for some $\alpha \in (0, 2]$. We refer to α as the index of F.

We say that F belongs to the domain of attraction of a distribution G if there exist constants $a_n > 0$, b_n such that $a_n^{-1}(S_n - b_n)$ converges in distribution to G as $n \to \infty$. This is clearly true for F if it is a stable distribution therefore any stable distribution posses a domain of attraction. In fact, a distribution is stable if and only if it possesses a domain of attraction. A useful result of [34, Chapter XVII.5] is that if F belongs to the domain of attraction of a stable law with index $\alpha < 2$ then ξ has finite absolute moments for all $\beta < \alpha$ and no moments of order $\beta > \alpha$ exist.

For much of the thesis we will consider only positive random variables so suppose ξ is almost surely positive. For $\zeta, \eta > 0$ let

$$U_\zeta(x) := \mathbb{E}\left[\xi^\zeta \mathbf{1}_{\{\xi \le x\}}\right] \qquad \text{and} \qquad V_\eta(x) := \mathbb{E}\left[\xi^\eta \mathbf{1}_{\{\xi \ge x\}}\right]$$

denote the truncated moment functions of ξ. By [34, Theorem IX.8.1], F belongs to the domain of attraction of a stable law with index $\alpha \in (0, 2]$ if and only if there exists a slowly varying function L such that, as $x \to \infty$,

$$U_2(x) \sim x^{2-\alpha} L(x). \tag{2.2}$$

Moreover, (2.2) is fully equivalent to

$$\mathbb{P}(\xi \ge x) \sim \frac{2-\alpha}{\alpha} x^{-\alpha} L(x) \tag{2.3}$$

when $\alpha < 2$ therefore we will often consider this relation instead.

Suppose that $\lim_{x\to\infty} U_\zeta(x) = \infty$; then, if either U_ζ or V_η varies regularly then, by [34, Theorem VIII.9.2] there exists a limit

$$\lim_{x\to\infty} \frac{x^{\zeta-\eta} V_\eta(x)}{U_\zeta(x)} = c \tag{2.4}$$

where c may be 0 or ∞; however, $c \in \{0, \infty\}$ can only be the case when U_ζ or V_η varies slowly.

This concludes the basic theory of stable laws that we require for the thesis. We now include two technical results which will be used later. Lemma 2.2.1 shows that the product of an exponential random variable with a heavy tailed random variable has a similar tail to the heavy tailed variable.

Lemma 2.2.1. *Let $X \sim exp(\theta)$ for $\theta > 0$ and suppose ξ is an independent, positive random variable which belongs to the domain of attraction of a stable law of index $\alpha \in (0, 2)$. Then $\mathbf{P}(X\xi > x) \sim \theta^{-\alpha}\Gamma(\alpha + 1)\mathbf{P}(\xi > x)$ as $x \to \infty$.*

Proof. For some slowly varying function L we have that $\mathbf{P}(\xi \geq x) \sim x^{-\alpha}L(x)$ as $x \to \infty$.

Fix $0 < u < 1 < v < \infty$ then $\forall y \leq u$ we have that $x/y > x$ thus $\mathbf{P}(\xi \geq x/y) \leq \mathbf{P}(\xi \geq x)$ it therefore follows that

$$0 \leq \int_0^u \theta e^{-\theta y} \frac{\mathbf{P}(\xi \geq x/y)}{\mathbf{P}(\xi \geq x)} \mathrm{d}y \leq \int_0^u \theta e^{-\theta y} \mathrm{d}y = 1 - e^{\theta u}.$$

We have that $\mathbf{P}(\xi \geq x/y)/\mathbf{P}(\xi \geq x) \to y^\alpha$ uniformly over $y \in [u, v]$ therefore

$$\lim_{x\to\infty} \int_u^v \theta e^{-\theta y} \frac{\mathbf{P}(\xi \geq x/y)}{\mathbf{P}(\xi \geq x)} \mathrm{d}y = \int_u^v \theta e^{-\theta y} y^\alpha \mathrm{d}y.$$

Moreover, since this holds for all $u \geq 0$ and $1 - e^{\theta u} \to 0$ as $u \to 0$ we have that

$$\lim_{x\to\infty} \int_0^v \theta e^{-\theta y} \frac{\mathbf{P}(\xi \geq x/y)}{\mathbf{P}(\xi \geq x)} \mathrm{d}y = \int_0^v \theta e^{-\theta y} y^\alpha \mathrm{d}y. \tag{2.5}$$

Since $0 < \mathbf{P}(\xi \geq x) \leq 1$ for all $x < \infty$ we have that L is bounded away from $\{0, \infty\}$ on any compact interval thus satisfies the requirements of Potter's theorem (see, for example, [19, Section 1.5.4]) that if L is slowly varying and bounded away from $\{0, \infty\}$ on any compact subset of $[0, \infty)$ then for any $\epsilon > 0$ there exists $A_\epsilon > 1$ such that for $x, y > 0$

$$\frac{L(z)}{L(x)} \leq A_\epsilon \max\left\{\left(\frac{z}{x}\right)^\epsilon, \left(\frac{x}{z}\right)^\epsilon\right\}.$$

Moreover, $\exists c_1, c_2 > 0$ such that $c_1 t^{-\alpha} L(t) \leq \mathbf{P}(\xi \geq t) \leq c_2 t^{-\alpha} L(t)$ hence we have that for all $y > v$ $\mathbf{P}(\xi \geq x/y)/\mathbf{P}(\xi \geq x) \leq C y^{\alpha+\epsilon}$. By dominated convergence we therefore have that

$$\lim_{x\to\infty} \int_v^\infty \theta e^{-\theta y} \frac{\mathbf{P}(\xi \geq x/y)}{\mathbf{P}(\xi \geq x)} \mathrm{d}y = \int_v^\infty \theta e^{-\theta y} y^\alpha \mathrm{d}y.$$

Combining this with (2.5) we have that

$$\lim_{x\to\infty} \frac{\mathbf{P}(X\xi \geq x)}{\mathbf{P}(\xi \geq x)} = \lim_{x\to\infty} \int_0^\infty \theta e^{-\theta y} \frac{\mathbf{P}(\xi \geq x/y)}{\mathbf{P}(\xi \geq x)} \mathrm{d}y = \int_0^\infty \theta e^{-\theta y} y^\alpha \mathrm{d}y = \theta^{-\alpha}\Gamma(\alpha + 1).$$

□

The following lemma concerning the form of the probability generating function of the offspring distribution will be fundamental in determining the distribution over the number of large traps rooted at a given backbone vertex in Chapter 4. The case $\mu = 1$ appears in [20]; the proof of Lemma 2.2.2 is a simple extension therefore we omit it.

Lemma 2.2.2. *Suppose the offspring distribution belongs to the domain of attraction of a stable law with index $\alpha \in (1, 2)$ and mean $\mathbf{E}[\xi] = \mu$.*

1. *If $\mu \leq 1$ then as $s \to 1^-$*

$$\mathbf{E}[s^\xi] - s^\mu \sim \frac{\Gamma(3-\alpha)}{\alpha(\alpha-1)}(1-s)^\alpha L((1-s)^{-1})$$

where $\Gamma(t) = \int_0^\infty x^{t-1}e^{-x}\mathrm{d}x$ is the usual gamma function.

2. *If $\mu > 1$ then*

$$1 - \mathbf{E}[s^\xi] = \mu(1-s) + \frac{\Gamma(3-\alpha)}{\alpha(\alpha-1)}(1-s)^\alpha \overline{L}((1-s)^{-1})$$

where $\overline{L}$ varies slowly at ∞.

2.3 Random walks and random variables

We now state several classical results for random variables which will be used throughout the thesis. Suppose that S_n is the partial sum of a sequence of independent, centred random variables X_k and $\lambda > 0$, then Kolmogorov's maximal inequality (e.g. [17]) states that

$$\mathbb{P}\left(\max_{1\leq k\leq n} |S_k| \geq \lambda\right) \leq \lambda^{-2}\sum_{k=1}^n \mathrm{Var}(X_k).$$

A similar result that we will use later is Doob's inequality (e.g. [33]) which states that if M_n is a submartingale then

$$\mathbb{P}\left(\max_{1\leq k\leq n} M_k \geq \lambda\right) \leq \lambda^{-1}\mathbb{E}\left[M_n \mathbf{1}_{\{\max_{1\leq k\leq n} M_k \geq \lambda\}}\right] \leq \lambda^{-1}\mathbb{E}\left[M_n \vee 0\right].$$

Another related result is the L^p maximal inequality (e.g. [33]) which states that for M_n a submartingale and $1 < p < \infty$ we have that

$$\mathbb{E}\left[\max_{1\leq k\leq n} M_k^p\right] \leq \left(\frac{p}{p-1}\right)^p \mathbb{E}\left[M_n^p \mathbf{1}_{\{M_n \geq 0\}}\right].$$

Binomial random variables will play a key role in the decomposition of excursion times in trees. Suppose that B is binomially distributed with n trials of success probability $p \in (0,1)$. Let $\mu = np$ be the expected number of successes then the Chernoff bounds (e.g. [60]) state that for $\delta \in (0,1)$ we have that

$$\mathbb{P}\left(B \geq (1+\delta)\mu\right) \leq e^{-\frac{\delta^2\mu}{3}} \qquad \text{and} \qquad \mathbb{P}\left(B \leq (1-\delta)\mu\right) \leq e^{-\frac{\delta^2\mu}{2}}.$$

For much of the thesis we will be concerned with invariance principles. To this end, we will often apply the quintessential Donsker's invariance principle (e.g. [32], [33]) which states that if S_n is the partial sum of a sequence of independent random variables X_k with mean 0 and variance 1 then $S_{\lfloor nt \rfloor} n^{-1/2}$ converges in distribution on $D_{J_1}([0,\infty),\mathbb{R})$ to a standard Brownian motion.

A useful technique for deriving limiting results is to exploit ergodicity. We will benefit from Birkhoff's ergodic theorem (e.g. [16], [48]) which gives us that if ξ is a random variable with law $\mathbb{P}$ in a space Ω and θ is an ergodic $\mathbb{P}$-preserving transformation on Ω then, for any measurable function $f \geq 0$, we have that $n^{-1}\sum_{k=0}^{n-1} f(\theta^k \xi)$ converges to $\mathbb{E}[f(\xi)]$ as $n \to \infty$ for $\mathbb{P}$-a.e. ξ.

The following result is [10, Theorem 10.2], and is itself a consequence of [66, Theorem IV.6]. This gives a set of necessary and sufficient conditions for a random i.i.d. sum to converge to a certain infinitely divisible law.

Proposition 2.3.1. *Let $n(t) : [0,\infty) \to \mathbb{N}$ and for each t let $\{R_k(t)\}_{k=1}^{n(t)}$ be a sequence of i.i.d. random variables. Assume that for every $\epsilon > 0$ it is true that*

$$\lim_{t\to\infty} \mathbb{P}(R_1(t) > \epsilon) = 0.$$

Now, suppose $\mathscr{L}(x) : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is a real, non-decreasing function satisfying $\lim_{x\to\infty} \mathscr{L}(x) = 0$ and $\int_0^a x^2 \mathrm{d}\mathscr{L}(x) < \infty$ for all $a > 0$. Let $d \in \mathbb{R}$ and $\varsigma \geq 0$, then the following statements are equivalent:

1. *As $t \to \infty$*

$$\sum_{k=1}^{n(t)} R_k(t) \xrightarrow{d} R_{d,\varsigma,\mathscr{L}}$$

where $R_{d,\varsigma,\mathscr{L}}$ has the law $\mathcal{I}(d,\varsigma,\mathscr{L})$, that is,

$$\mathbb{E}[e^{itR_{d,\varsigma,\mathscr{L}}}] := \exp\left(idt - \frac{\varsigma^2 t^2}{2} + \int_{-\infty}^{\infty} \left(e^{itx} - 1 - \frac{itx}{1+x^2} \right) \mathrm{d}\mathscr{L}(x) \right). \tag{2.6}$$

2. *For $\tau > 0$ let $\overline{R}_\tau(t) := R_1(t)\mathbf{1}_{\{|R_1(t)|\leq\tau\}}$ then for every continuity point x of $\mathscr{L}$*

$$\begin{aligned} d &= \lim_{t\to\infty} n(t)\mathbb{E}[\overline{R}_\tau(t)] + \int_{|x|>\tau} \frac{x}{1+x^2} \mathrm{d}\mathscr{L}(x) - \int_{0<|x|\leq\tau} \frac{x^3}{1+x^2} \mathrm{d}\mathscr{L}(x), \\ \varsigma^2 &= \lim_{\tau\to 0} \limsup_{t\to\infty} n(t)\mathrm{Var}(\overline{R}_\tau(t)), \\ \mathscr{L}(x) &= \begin{cases} \lim_{t\to\infty} n(t)\mathbb{P}(R_1(t) \leq x) & x < 0 \\ -\lim_{t\to\infty} n(t)\mathbb{P}(R_1(t) > x) & x > 0 \end{cases} \end{aligned}$$

A fundamental aspect of the random walk on GW-tree model is the concept of trapping. For this reason it will be extremely important to understand the time spent by the walk in finite trees. Let $\mathscr{T}$ be a fixed, rooted tree with n^{th} generation size $Z_n^{\mathscr{T}}$.

For a β-biased random walk X on $\mathscr{T}$ and $x \in \mathscr{T}$ let $\tau_x^+ := \inf\{m > 0 : X_m = x\}$ denote the first return time to x. It is classical (e.g. [56]) that, when $Z_1^{\mathscr{T}} \geq 1$,

$$E_\rho^{\mathscr{T}}[\tau_\rho^+] = 2\sum_{n\geq 1} \frac{Z_n^{\mathscr{T}}\beta^{n-1}}{Z_1^{\mathscr{T}}}. \tag{2.7}$$

Write $\tau_n^Y := \inf\{m \geq 0 : \ Y_m = n\}$ to be the first hitting time of level n by β-biased walk Y on $\mathbb{Z}$. The following lemma describes the probability that the embedded walk moves back k levels before moving forward n. This is the classical Gambler's ruin (see, for example, [71]) therefore we omit the proof.

Lemma 2.3.2. *For integers $k < 0 < n$*

$$P_0(\tau_k^Y < \tau_n^Y) = \frac{\beta^n - 1}{\beta^{n-k} - 1}.$$

We can deduce from this lemma that the probability, started from the origin, that the ($\beta > 1$) biased walk never reaches vertex $k < 0$ is

$$P_0(\tau_k^Y < \infty) = \lim_{n\to\infty} P_0(\tau_k^Y < \tau_n^Y) = \beta^{-k}.$$

In particular, the escape probability is given by $P_0(\tau_{-1}^+ = \infty) = 1 - \beta^{-1}$.

Starting from a vertex on the backbone of an infinite tree, we will want to know how many times the walk takes an excursion into a finite tree attached to the starting vertex before escaping. Let $\mathcal{T}$ be a subcritical GW-tree conditioned to survive, $x \in \mathcal{Y} \setminus \{\rho\}$ be vertex on the backbone which is not the root and A be a non-empty subset of $c(x)\setminus\mathcal{Y}$. For a β-biased walk X on $\mathcal{T}$ and $y \in A$ let $W^y := |\{m > 0 : X_{m-1} = x, X_m = y\}|$ be the total number of times y is reached from x. The following lemma describes the number of visits to traps before the walk moves along the backbone.

Lemma 2.3.3. *Under $P^{\mathcal{T}}$ we have that*

$$\sum_{y\in A} W^y \sim Geo\left(\frac{\beta - 1}{(|A|+1)\beta - 1}\right)$$

and $(W^y)_{y\in A}$ have a negative multinomial distribution with one failure until termination and probabilities

$$q_y := \begin{cases} \frac{\beta-1}{(|A|+1)\beta-1} & y = \rho \\ \frac{\beta}{(|A|+1)\beta-1} & y \in A \end{cases}$$

that from x the next excursion will be into the trap at y (where $y = \rho$ denotes escaping).

Proof. Let $\overrightarrow{x}$ denote the child of x on the backbone. From y the walk must return to x before escaping therefore any traps not rooted in A can be ignored and it suffices to assume that $|c(x)| = k+1$ and $A = \{y_1, ..., y_k\}$. Similarly, since the walk cannot escape from the parent of x before first reaching x we can ignore this vertex. By comparison with a biased random walk on $\mathbb{Z}$ we have that $P_{\overrightarrow{x}}(\tau_x^+ = \infty) = 1 - \beta^{-1}$. Since the walk moves to any child of x with the same probability we have that $P_x(\tau_z^+ = \min_{y \in c(x)} \tau_y^+) = (k+1)^{-1}$ for any $z \in c(x)$. The probability of never entering a trap in the branch at x is, therefore,

$$P_x\left(\bigcap_{j=1}^{k}\{\tau_{y_j}^+ = \infty\}\right) = \sum_{l=0}^{\infty}\left(\frac{1}{k+1}\beta^{-1}\right)^l\left(\frac{1-\beta^{-1}}{k+1}\right) = \frac{\beta-1}{(k+1)\beta-1}.$$

Each excursion ends with the walker at x thus the walk takes a geometric number of excursions into traps with escape probability $(\beta-1)/((k+1)\beta-1)$. The second statement then follows from the fact that the walker has equal probability of going into any of the traps. $\square$

Lemma 2.3.2 also gives us the following corollary which determines the probabilities of reaching the deepest point in a trap, escaping the trap from the deepest point and the transition probabilities for the walk in the trap conditional on reaching the deepest point before escaping. For a tree $\mathscr{T}$ of height $\mathscr{H}$, root ρ and deepest vertex δ denote by $\delta_0 = \rho, \delta_1, ..., \delta_{\mathscr{H}} = \delta$ the unique self avoiding path from ρ to δ.

Lemma 2.3.4. *For any tree $\mathscr{T}$ of height $\mathscr{H}$ (with $\mathscr{H} \geq 2$), root ρ and deepest vertex δ we have that:*

$$P_{\delta_1}^{\mathscr{T}}(\tau_\delta^+ < \tau_\rho^+) = \frac{1-\beta^{-1}}{1-\beta^{-\mathscr{H}}}$$

is the probability of reaching the deepest point without escaping;

$$P_\delta^{\mathscr{T}}(\tau_\rho^+ < \tau_\delta^+) = \frac{\beta-1}{\beta^{\mathscr{H}}-1}$$

is the probability of escaping from the deepest point before returning;

$$P_{\delta_k}^{\mathscr{T}}(\tau_{\delta_{k-1}}^+ < \tau_{\delta_{k+1}}^+ | \tau_\delta^+ < \tau_\rho^+) = \frac{1-\beta^{-(\mathscr{H}+1-k)}}{1-\beta^{-(\mathscr{H}-k)}} \cdot \frac{\beta}{\beta+1}$$

is the probability that the walk restricted to the spine conditioned on reaching δ before returning to ρ moves towards δ.

We now show an important identity that will allow us to prove an upper bound on the correlation between the number of visits to two vertices in a tree. In order to show this we will use a decomposition which counts the number of visits to each vertex. For $z_1, z_2, z_3 \in \mathscr{T}$ write

$$q_{z_1}(z_2, z_3) := P_{z_1}^{\mathscr{T}}(\tau_{z_2}^+ < \tau_{z_3}^+)$$

to be the probability that the walk started from z_1 hits z_2 before z_3. We also require a similar expression for a walk on a tree. Let $\mathscr{T}_{x,y}$ denote a tree with root ρ in which every vertex has a single offspring except the vertices w, x, y where w has two offspring and x, y have none. Denote these offspring w_x, w_y then let x, y be descendants of w_x, w_y respectively (possibly w_x, w_y). For vertices z_1, z_2, z_3, z_4 write

$$q_{z_1}(z_2, \{z_3, z_4\}) := P_{z_1}^{\mathscr{T}_{x,y}}(\tau_{z_2}^+ < \tau_{z_3}^+ \wedge \tau_{z_4}^+)$$

as the probability that the walk started at z_1 reaches z_2 before z_3 or z_4 by a β-biased walk on $\mathscr{T}_{x,y}$.

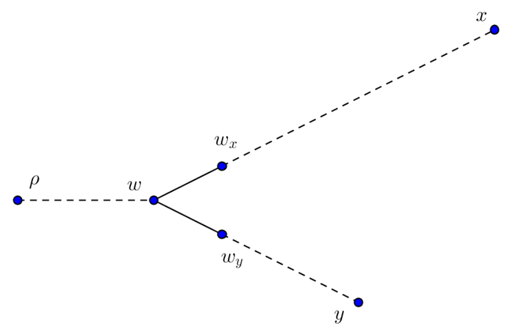

Figure 2.1: The tree $\mathscr{T}_{x,y}$ with single branching point w and extremal points ρ, x, y.

Lemma 2.3.5 gives the probability that the walk started at w reaches ρ before x or y. Alternatively, this can be shown by comparing with an electrical network with conductances β^k between vertices in generations $k, k+1$ and then using network reduction. See, for example, [56, Chapter 2] for a more detailed explanation of network reduction and the connections between random walks and electrical networks.

Lemma 2.3.5. *For any $\mathscr{T}_{x,y}$,*

$$q_w(\rho, \{x, y\}) = \frac{(\beta^{|y|-|w|} - 1)(\beta^{|x|-|w|} - 1)}{2\beta^{|y|+|x|-|w|} - \beta^{|y|+|x|-2|w|} - \beta^{|x|} - \beta^{|y|} + 1}.$$

Proof. Write w_ρ as the parent of w then

$$\begin{aligned}
q_w(\rho, \{x, y\}) &= \frac{1}{2\beta+1} q_{w_\rho}(\rho, \{x, y\}) + \frac{\beta}{2\beta+1} q_{w_x}(\rho, \{x, y\}) + \frac{\beta}{2\beta+1} q_{w_y}(\rho, \{x, y\}) \\
q_{w_\rho}(\rho, \{x, y\}) &= q_w(\rho, \{x, y\}) q_{w_\rho}(w, \rho) + q_{w_\rho}(\rho, w) \\
q_{w_x}(\rho, \{x, y\}) &= q_w(\rho, \{x, y\}) q_{w_x}(w, x) \\
q_{w_y}(\rho, \{x, y\}) &= q_w(\rho, \{x, y\}) q_{w_y}(w, y).
\end{aligned}$$

Combining these gives us that

$$\begin{aligned}
q_w(\rho,\{x,y\}) &= \frac{q_w(\rho,\{x,y\})}{2\beta+1}\left(q_{w_\rho}(w,\rho)+\beta q_{w_x}(w,x)+\beta q_{w_y}(w,y)\right)+\frac{q_{w_\rho}(\rho,w)}{2\beta+1}\\
&= \frac{q_{w_\rho}(\rho,w)}{2\beta+q_{w_\rho}(\rho,w)-q_{w_x}(w,x)-q_{w_y}(w,y)}\\
&= \frac{\frac{\beta-1}{\beta^{|w|}-1}}{2\beta+\frac{\beta-1}{\beta^{|w|}-1}-\frac{\beta^{|x|-|w|}-\beta}{\beta^{|x|-|w|}-1}-\frac{\beta^{|y|-|w|}-\beta}{\beta^{|y|-|w|}-1}}
\end{aligned}$$

by Lemma 2.3.2. Rearranging gives the result. $\square$

Let $\mathscr{T}$ be a fixed tree and $(X_n)_{n\geq 1}$ a β-biased walk on $\mathscr{T}$. For $x\in\mathscr{T}$ let

$$v_x := \sum_{k=1}^{\tau_\rho^+}\mathbf{1}_{\{X_k=x\}}$$

denote the number of visits to x before returning to ρ. Then $\tau_\rho^+=\sum_{x\in\mathscr{T}}v_x$ and

$$E_\rho^{\mathscr{T}}\left[(\tau_\rho^+)^2\right]=\sum_{x,y\in\mathscr{T}}E_\rho^{\mathscr{T}}[v_xv_y].$$

For any $x,y\in\mathscr{T}$ there exists a unique vertex $w_{x,y}$ which is the closest ancestor of both x and y. We will often write w instead of $w_{x,y}$ when it is clear to which vertices we are referring. Moreover

$$E_\rho^{\mathscr{T}}[v_xv_y]=P_\rho^{\mathscr{T}}(\tau_{w_{x,y}}^+<\tau_\rho^+)E_{w_{x,y}}^{\mathscr{T}}[v_xv_y]$$

where, by comparison with a simple biased random walk on $\mathbb{Z}$, we have that

$$P^{\mathscr{T}}(\tau_{w_{x,y}}^+<\tau_\rho^+)\in[1-\beta^{-1},1].$$

We now prove a bound on $E_w^{\mathscr{T}}[v_xv_y]$ following a similar method to that used in [49] for the unbiased case. Recall that $c(x)$ is the set of children of x in $\mathscr{T}$.

Lemma 2.3.6. *For $\beta>1$, there exists a constant C_β such that for any finite tree $\mathscr{T}$,*

$$E_\rho^{\mathscr{T}}[v_xv_y]\leq E_w^{\mathscr{T}}[v_xv_y]\leq C_\beta(|c(x)|\beta+1)(|c(y)|\beta+1)\beta^{|x|+|y|}.$$

Proof. When $w=\rho$ at least one of x and y is never reached therefore $v_xv_y=0$ and we may assume $|w|\geq 1$. There are now three cases to consider; these are:

1. $x=y=w_{x,y}$;

2. $x=w_{x,y}\neq y$;

3. $x \neq w_{x,y} \neq y$.

In case 1 we have that v_x is geometrically distributed with termination probability $q_x(\rho, x)$ therefore

$$E^{\mathscr{T}}_{w_{x,y}}[v_x v_y] = E^{\mathscr{T}}_x[v_x^2] = \frac{q_x(x,\rho)+1}{q_x(\rho,x)^2}.$$

For $x \notin c(\rho)$ we have that $\beta/(1+\beta) \leq q_x(x,\rho) \leq 1$ and by Lemma 2.3.2

$$q_x(\rho, x) = \frac{1-\beta^{-1}}{(|c(x)|\beta+1)(\beta^{|x|-1}-\beta^{-1})}.$$

We therefore have that

$$E^{\mathscr{T}}_x[v_x^2] \leq C_\beta(|c(x)|\beta+1)^2\beta^{2|x|}.$$

In case 2, the number of visits to x from x is geometrically distributed as in case 1. For each visit to x (except the last) the walk reaches y before returning to x with probability $q_x(y,x)/q_x(x,\rho)$ since, due to the tree structure, the walk cannot move from ρ to y without hitting x. From y, the walk returns to y a geometric number of times before returning to x. More specifically,

$$E^{\mathscr{T}}_{w_{x,y}}[v_x v_y] = E^{\mathscr{T}}_x[v_x v_y] = \sum_{j=1}^{\infty} j q_x(\rho,x) q_x(x,\rho)^{j-1} E^{\mathscr{T}}_x[v_y | v_x = j]$$

where, conditional on the event $\{v_x = j\}$, we have that v_y is equal in distribution to the sum of $B^j_{x,y} \sim Bin(j-1, q_x(y,x)/q_x(x,\rho))$ independent geometric random variables $G^i_{x,y} \sim Geo(q_y(x,y))$. Under $P^{\mathscr{T}}$ the number of excursions are independent therefore

$$E^{\mathscr{T}}_x[v_y|v_x = j] = (j-1)\frac{q_x(y,x)}{q_x(x,\rho)} \cdot \frac{1}{q_y(x,y)}.$$

We therefore have that

$$\begin{aligned} E^{\mathscr{T}}_{w_{x,y}}[v_x v_y] &= \frac{q_x(y,x)q_x(\rho,x)}{q_x(x,\rho)q_y(x,y)} \sum_{j=1}^{\infty} j(j-1)q_x(x,\rho)^{j-1} \\ &= \frac{q_x(y,x)q_x(\rho,x)}{q_x(x,\rho)q_y(x,y)} \cdot \frac{2q_x(x,\rho)}{q_x(\rho,x)^3} \\ &= \frac{2q_x(y,x)}{q_y(x,y)q_x(\rho,x)^2}. \end{aligned} \tag{2.8}$$

Using Lemma 2.3.2 we then have that

$$q_x(y,x) = \frac{\beta}{|c(x)|\beta+1} \cdot \frac{1-\beta^{-1}}{1-\beta^{|x|-|y|}},$$

$$
\begin{aligned}
q_y(x,y) &= \frac{1}{|c(y)|\beta+1}\cdot\frac{\beta-1}{\beta^{|y|-|x|}-1},\\
q_x(\rho,x) &= \frac{1}{|c(x)|\beta+1}\cdot\frac{\beta-1}{\beta^{|x|}-1}.
\end{aligned}
$$

Combining these with (2.8) we have that

$$
E^{\mathscr{T}}_{w_{x,y}}[v_x v_y] \le C_\beta(|c(x)|\beta+1)(|c(y)|\beta+1)\beta^{|x|+|y|}.
$$

In case 3, started from $w_{x,y}$, the walk reaches either x or y before returning to ρ with probability $q_{w_{x,y}}(\{x,y\},\rho)$. From x the walk has a geometric number of returns to x before returning to $w_{x,y}$. Moreover, from x, the walk must return to $w_{x,y}$ before reaching either ρ or y by definition of $w_{x,y}$. The same also holds switching x and y. Letting

$$
\overline{q}_w(x,y) = P^{\mathscr{T}}_w\Big(\tau^+_x < \tau^+_y | \tau^+_{\{x,y\}} < \tau^+_\rho\Big) \quad \text{and} \quad \overline{q}_w(y,x) = P^{\mathscr{T}}_w\Big(\tau^+_y < \tau^+_x | \tau^+_{\{x,y\}} < \tau^+_\rho\Big)
$$

we then have that $E^{\mathscr{T}}_w[v_x v_y]$ is equal to

$$
\sum_{j=0}^{\infty} q_w(\{x,y\},\rho)^j q_w(\rho,\{x,y\}) \sum_{k=0}^{j} \overline{q}_w(x,y)^k \overline{q}_w(y,x)^{j-k}\binom{j}{k}\frac{k(j-k)}{q_x(w,x)q_y(w,y)}
$$

since $q_x(w,x)^{-1}$ is the expected number of visits to x (started from x) before returning to w (and similarly for y) which are independent. Rearranging gives

$$
\begin{aligned}
&\sum_{k=0}^{j} \overline{q}_w(x,y)^k \overline{q}_w(y,x)^{j-k}\binom{j}{k}\frac{k(j-k)}{q_x(w,x)q_y(w,y)}\\
&\qquad = \frac{1}{q_x(w,x)q_y(w,y)}\sum_{k=1}^{j-1}\overline{q}_w(x,y)^k \overline{q}_w(y,x)^{j-k}\frac{j!}{(k-1)!(j-k-1)!}\\
&\qquad = j(j-1)\frac{\overline{q}_w(x,y)\overline{q}_w(y,x)}{q_x(w,x)q_y(w,y)}\sum_{l=0}^{j-2}\overline{q}_w(x,y)^l \overline{q}_w(y,x)^{j-2-l}\frac{(j-2)!}{l!(j-2-l)!}\\
&\qquad = j(j-1)\frac{\overline{q}_w(x,y)\overline{q}_w(y,x)}{q_x(w,x)q_y(w,y)}.
\end{aligned}
$$

Substituting back into the above formula for $E^{\mathscr{T}}_w[v_x v_y]$ it follows that

$$
\begin{aligned}
E^{\mathscr{T}}_w[v_x v_y] &= \frac{\overline{q}_w(x,y)\overline{q}_w(y,x)q_w(\rho,\{x,y\})}{q_x(w,x)q_y(w,y)}\sum_{j=0}^{\infty} j(j-1)q_w(\{x,y\},\rho)^j\\
&= \frac{\overline{q}_w(x,y)\overline{q}_w(y,x)q_w(\rho,\{x,y\})}{q_x(w,x)q_y(w,y)}\cdot\frac{2q_w(\{x,y\},\rho)^2}{q_w(\rho,\{x,y\})^3}
\end{aligned}
$$

$$= \frac{2q_w(\{x,y\},\rho)^2\overline{q}_w(x,y)\overline{q}_w(y,x)}{q_w(\rho,\{x,y\})^2 q_x(w,x)q_y(w,y)}.$$

The terms in the numerator can all be bounded below by half of the escape probability $1-\beta^{-1}$ therefore we gain nothing using their exact expressions and bound them above by 1. Using Lemmas 2.3.2 and 2.3.5 for the other terms we have that

$$q_w(\rho,\{x,y\}) = \frac{(\beta^{|y|-|w|}-1)(\beta^{|x|-|w|}-1)}{2\beta^{|y|+|x|-|w|}-\beta^{|y|+|x|-2|w|}-\beta^{|x|}-\beta^{|y|}+1},$$
$$q_x(w,x) = \frac{1}{|c(x)|\beta+1}\cdot\frac{\beta-1}{\beta^{|x|-|w|}-1},$$
$$q_y(w,y) = \frac{1}{|c(y)|\beta+1}\cdot\frac{\beta-1}{\beta^{|y|-|w|}-1}.$$

Since $|y|\geq 1$ we have that $\beta^{|y|}\geq 1$ therefore

$$q_w(\rho,\{x,y\}) \geq \frac{(\beta^{|y|-|w|}-1)(\beta^{|x|-|w|}-1)}{2\beta^{|y|+|x|-|w|}}$$

and

$$E_w^{\mathscr{T}}[v_x v_y] \leq \frac{2}{q_w(\rho,\{x,y\})^2 q_x(w,x)q_y(w,y)} \leq C_\beta(|c(x)|\beta+1)(|c(y)|\beta+1)\beta^{|x|+|y|}.$$

□

We now state [30, Lemma 5.1] which will be an important result throughout this thesis. It shows that the regeneration times of a biased random walk on $\mathbb{Z}$ have exponential moments. Let Y_n be a β-biased random walk on $\mathbb{Z}$ started from 0. Define $\zeta_0^Y := 0$ and $\zeta_k^Y := \inf\{m > \zeta_{k-1}^Y : \{Y_n\}_{n=0}^{m-1}\cap\{Y_n\}_{n=m}^{\infty}=\emptyset\}$ for $k\geq 1$ be the successive regeneration times for the walk. We refer to the points $\varrho_k := Y_{\zeta_k^Y}$ as regeneration points.

Lemma 2.3.7. *If $\beta>1$ then there exists $s>1$ such that, for any $k\geq 0$,*

$$E[s^{\varrho_{k+1}-\varrho_k}], E[s^{\zeta_{k+1}^Y-\zeta_k^Y}] < \infty.$$

2.4 Branching processes

In this section we state several technical results concerning branching processes which are of key importance throughout this thesis when working with GW-trees. An important result for branching processes (see, for example [54]), is that if Z_n is a GW process with $\mu<1$ then the sequence $\mathbf{P}(Z_n>0)/\mu^n$ is decreasing; moreover, $\mathbf{E}[\xi\log(\xi)]<\infty$ if and only if the limit of $\mathbf{P}(Z_n>0)\mu^{-n}$ as $n\to\infty$ exists and is strictly positive. This assumption holds under any of the hypotheses assumed in this thesis therefore we will always make this assumption and let c_μ be the constant such that

$$\mathbf{P}(Z_n > 0) \sim c_\mu \mu^n. \tag{2.9}$$

This is particularly important because it will allow us to deduce the distribution of the heights of the branches attached to the backbone of a GW-tree. The heights of these trees are fundamental in the trapping phenomena observed in Chapter 4.

Lemma 2.4.1 shows bounds on the moments of the generation sizes.

Lemma 2.4.1. *Let Z_n denote the n^{th} generation size of an f-GW-process with offspring distribution ξ and mean $\mu \in (0,1)$.*

1. $\mathbf{E}[Z_n] = \mu^n$.
2. *If $\mathbf{E}[\xi^2] < \infty$ and $m \geq n$ then $c\mu^m \leq \mathbf{E}[Z_n Z_m] \leq C\mu^m$ for some constants c, C.*
3. *If $\mathbf{E}[\xi^3] < \infty$ and $l \geq m \geq n$ then $c\mu^l \leq \mathbf{E}[Z_n Z_m Z_l] \leq C\mu^l$ for some constant c, C.*

Proof. Let f_n denote the generating function of Z_n then statement 1 follows from

$$\mathbf{E}\left[Z_n\right] \;=\; \sum_{j=1}^{\infty} j\mathbf{P}\left(Z_n = j\right) \;=\; f_n'(1) \;=\; f_{n-1}'(1)f'(1) \;=\; f'(1)^n \;=\; \mu^n. \tag{2.10}$$

Since Z_n is integer valued and 0 is absorbing we have that $\mathbf{E}[Z_n Z_m] \geq \mathbf{P}(Z_m \geq 1) \geq c\mu^m$ and $\mathbf{E}[Z_n Z_m Z_l] \geq \mathbf{P}(Z_l \geq 1) \geq c\mu^l$ which proves the lower bounds.

If $\mathbf{P}(\xi < 2) = 1$ then Z_n only takes values 0 and 1. In particular, for $l \geq m \geq n$,

$$\mathbf{E}[Z_n Z_m Z_l] \;=\; \mathbf{E}[Z_m Z_l] \;=\; \mathbf{E}[Z_l] \;=\; \mu^l$$

therefore the results follow. We therefore assume that $\mathbf{P}(\xi \geq 2) > 0$ which implies that $f''(1) > 0$.

For the second moment we have that $\mathbf{E}[Z_n^2] = f_n''(1) + \mathbf{E}[Z_n]$ where

$$f_{n+1}''(1) \;=\; \left(f\left(f_n(s)\right)'\right)'\Big|_{s=1} \;=\; f''(1)\mu^{2n} + \mu f_n''(1).$$

Applying this recursively we see that

$$f_{n+1}''(1) \;=\; f''(1)\sum_{k=0}^{n} \mu^{n+k} \;=\; f''(1)\mu^n \frac{1-\mu^{n+1}}{1-\mu}$$

therefore

$$\mathbf{E}[Z_n^2] = c_1\mu^n + c_2\mu^{2n} \tag{2.11}$$

whenever $f''(1) < \infty$ which follows from $\mathbf{E}[\xi^2] < \infty$. For $m > n$, by stationarity of GW-processes and (2.10) we have that

$$\mathbf{E}[Z_m|Z_n = k] \;=\; \mathbf{E}[Z_{m-n}|Z_0 = k] \;=\; k\mathbf{E}[Z_{m-n}] \;=\; k\mu^{m-n}.$$

Therefore, statement 2 follows by

$$\begin{aligned}\mathbf{E}[Z_n Z_m] &= \sum_{k=1}^{\infty} k\mathbf{P}(Z_n = k)\mathbf{E}[Z_m|Z_n = k] \\ &= \sum_{k=1}^{\infty} k^2\mathbf{P}(Z_n = k)\mu^{m-n} \\ &= \mu^{m-n}\mathbf{E}[Z_n^2] \\ &\le C\mu^m.\end{aligned}$$

When $\mathbf{E}[\xi^3] < \infty$ we have that $f''(1)$, $f'''(1) < \infty$ therefore differentiating $f_n''(s)$ and evaluating at $s = 1$ gives us that

$$\begin{aligned}f_{n+1}'''(1) &= 3f''(1)f_n'(1)f_n''(1) + f_n'(1)^3 f'''(1) + f'(1)f_n'''(1) \\ &= \left(f'''(1) - \frac{3f''(1)^2}{\mu(1-\mu)}\right)\mu^{3n} + \frac{3f''(1)^2}{\mu(1-\mu)}\mu^{2n} + \mu f_n'''(1).\end{aligned}$$

Iterating gives us that

$$f_{n+1}'''(1) = \left(f'''(1) - \frac{3f''(1)^2}{\mu(1-\mu)}\right)\frac{1-\mu^{2n}}{1-\mu}\mu^{3n} + \frac{3f''(1)^2}{\mu(1-\mu)} \cdot \frac{1-\mu^n}{1-\mu}\mu^n + \mu^n f'''(1)$$

which proves that, for some $c_1 > 0$,

$$\mathbf{E}[Z_n^3] = c_1\mu^n + c_2\mu^{2n} + c_3\mu^{3n} + c_4\mu^{5n} \tag{2.12}$$

If $l \ge m \ge n$ and $\mathbf{E}[\xi^3] < \infty$ then for any $j, k \ge 1$ (where $j = k$ if $m = n$)

$$\mathbf{E}[Z_n Z_m Z_l|Z_m = j, Z_n = k] \;=\; kj\mathbf{E}[Z_l|Z_m = j] \;=\; kj^2\mathbf{E}[Z_{l-m}] \;=\; kj^2\mu^{l-m}$$

by stationarity and (2.10) therefore

$$\begin{aligned}\mathbf{E}[Z_n Z_m Z_l] &= \sum_{k=1}^{\infty}\mathbf{P}(Z_n = k)\sum_{j=1}^{\infty}\mathbf{P}(Z_m = j|Z_n = k)\mathbf{E}[Z_n Z_m Z_l|Z_m = j, Z_n = k] \\ &= \mu^{l-m}\sum_{k=1}^{\infty} k\mathbf{P}(Z_n = k)\sum_{j=1}^{\infty} j^2\mathbf{P}(Z_m = j|Z_n = k) \\ &= \mu^{l-m}\sum_{k=1}^{\infty} k\mathbf{P}(Z_n = k)\mathbf{E}[Z_m^2|Z_n = k].\end{aligned} \tag{2.13}$$

For $j \geq 1$ let $Z_n^{(j)}$ be independent copies of Z_n then by convexity and (2.11)

$$\mathbf{E}[Z_m^2|Z_n = k] \;=\; k^2\mathbf{E}\left[\left(\sum_{j=1}^{k}\frac{Z_{m-n}^{(j)}}{k}\right)^2\right] \;\leq\; k^2\mathbf{E}[Z_{m-n}^2] \;\leq Ck^2\mu^{m-n}$$

therefore, combining this with (2.13) we have that

$$\mathbf{E}[Z_nZ_mZ_l] \;\leq\; C\mu^{l-n}\mathbf{E}[Z_n^3] \;\leq\; C\mu^l,$$

where the final inequality follows by (2.12), which gives statement 3.

□

Let $\overline{\mathcal{T}}$ be a subcritical GW-tree $\mathcal{T}$ rooted at ρ with an additional vertex $\overline{\rho}$ appended as the parent of ρ. By the renewal property of branching processes, we have that this is equal in distribution to a GW-tree rooted at $\overline{\rho}$ conditioned to have a single vertex in the first generation. From (2.7) it follows that

$$E_\rho^{\overline{\mathcal{T}}}[\tau_{\overline{\rho}}^+] \;=\; E_{\overline{\rho}}^{\overline{\mathcal{T}}}[\tau_{\overline{\rho}}^+] - 1 \;=\; 2\sum_{n\geq 0}Z_n^{\mathcal{T}}\beta^n - 1.$$

Recall that $\mathcal{H}(\mathscr{T})$ denotes the height of a tree $\mathscr{T}$. For any $m \geq 1$ we have that $\mathbf{P}(\mathcal{H}(\overline{\mathcal{T}}) \leq m) \geq p_0$ therefore, for some constant C,

$$\mathbf{E}\left[E_\rho^{\overline{\mathcal{T}}}[\tau_{\overline{\rho}}^+]|\mathcal{H}(\overline{\mathcal{T}}) \leq m\right] \;\leq\; \frac{\mathbf{E}\left[2\sum_{n=0}^{m-1}Z_n^{\overline{\mathcal{T}}}\beta^n - 1\right]}{\mathbf{P}(\mathcal{H}(\overline{\mathcal{T}}) \leq m)} \;\leq\; \begin{cases} C(\mu\beta)^m & \beta\mu > 1 \\ Cm & \beta\mu = 1 \\ C & \beta\mu < 1. \end{cases} \tag{2.14}$$

Lemma 2.4.2. *Let Z_n be a subcritical Galton-Watson process with mean μ and offspring ξ satisfying $\mathbf{E}[\xi^{1+\tilde{\varepsilon}}] < \infty$ for some $\tilde{\varepsilon} > 0$. Suppose $1 < \beta < \mu^{-1}$, then there exists $\kappa > 0$ such that for all $\epsilon \in (0, \kappa)$ we have that $(Z_n\beta^n)^{1+\epsilon}$ is a supermartingale.*

Proof. Let $\mathcal{F}_n := \sigma(Z_k;\; k \leq n)$ denote the natural filtration of Z_n and $(\xi_k)_{k\geq 1}$ be independent copies of ξ.

$$\begin{aligned}\mathbf{E}[(Z_n\beta^n)^{1+\epsilon}|\mathcal{F}_{n-1}] &= (Z_{n-1}\beta^{n-1})^{1+\epsilon}\beta^{1+\epsilon}\mathbf{E}\left[\left(\sum_{k=1}^{Z_{n-1}}\frac{\xi_k}{Z_{n-1}}\right)^{1+\epsilon}\Big|Z_{n-1}\right] \\ &\leq (Z_{n-1}\beta^{n-1})^{1+\epsilon}\beta^{1+\epsilon}\mathbf{E}\left[\sum_{k=1}^{Z_{n-1}}\frac{\xi_k^{1+\epsilon}}{Z_{n-1}}\Big|Z_{n-1}\right] \\ &= (Z_{n-1}\beta^{n-1})^{1+\epsilon}\beta^{1+\epsilon}\mathbf{E}[\xi^{1+\epsilon}]\end{aligned}$$

where the inequality follows by convexity of $f(x) = x^{1+\epsilon}$. From this it follows that for $\epsilon \in (0, \alpha - 1)$

$$\mathbf{E}[(Z_n\beta)^{1+\epsilon}] \;\leq\; \mathbf{E}[(Z_{n-1}\beta)^{1+\epsilon}]\mathbf{E}[(\xi\beta)^{1+\epsilon}] \;\leq\; \mathbf{E}[(\xi\beta)^{1+\epsilon}]^n \;<\; \infty.$$

Fix $\lambda = (\mu/\beta)^{1/2}$ then $\mu < \lambda$ and for $\epsilon > 0$ sufficiently small $\lambda\beta^{1+\epsilon} < 1$. By dominated convergence $\mathbf{E}[\xi^{1+\epsilon}] < \lambda$ for all ϵ small. In particular, $\beta^{1+\epsilon}\mathbf{E}[\xi^{1+\epsilon}] < 1$ for ϵ suitably small and therefore $(Z_n\beta^n)^{1+\epsilon}$ is a supermartingale. □

2.5 Describing the walk on the subcritical tree as a randomly trapped random walk

A subcritical GW-tree conditioned to survive consists of a semi-infinite path emanating from a fixed root with random finite trees attached to each vertex on the path. Each of these finite trees is typically quite short therefore the walk does not deviate too far from the origin. The biased randomly trapped random walk is transient and therefore does not spend much time at vertices below the origin. In this section we make these two statements more precise in order to show that we can couple the walk on the tree to a randomly trapped random walk in such a way that the two walks do not deviate too far from each other.

For this section we consider X_n to be the β-biased random walk on a subcritical GW-tree conditioned to survive $\mathcal{T}$. Recall that for a fixed tree $\mathscr{T}$ with root ρ we write $\mathcal{H}(\mathscr{T}) := \max_{x\in\mathscr{T}} d(\rho, x)$ to be the height of the tree. Let $\tilde{X}_n$ be the projection of X_n onto $\mathcal{Y}$; that is, $\tilde{X}_n$ is the unique vertex on $\mathcal{Y}$ which satisfies $d(X_n, \tilde{X}_n) = \min_{y\in\mathcal{Y}} d(X_n, y)$. Lemma 2.5.1 shows that the walk never deviates too far from the backbone.

Lemma 2.5.1. *Suppose $\mu \in (0,1)$ and $\beta \geq 1$ then $\mathbb{P}$-a.s.*

$$\sup_{n\geq 2}\sup_{m\leq nT} \frac{d(X_m, \tilde{X}_m)}{\log(n)} < \infty.$$

Proof. Recall that $\mathcal{T}^f$ denotes an f-GW tree. The distance $d(X_m, \tilde{X}_m)$ between the walk and the backbone is at most the height of the largest branch seen up to time nT therefore, since the walk can have visited at most M buds by time M, by a union bound we have that for $C > 0$

$$\mathbb{P}\left(\sup_{m\leq nT} d(X_m, \tilde{X}_m) > C\log(n)\right) \leq \lceil nT\rceil \mathbf{P}\left(\mathcal{H}(\mathcal{T}^f) > C\log(n) - 1\right).$$

Therefore, by (2.9),

$$\lceil nT \rceil \mathbf{P}\left(\mathcal{H}(\mathcal{T}^f) > C\log(n)\right) \leq C_T n \mu^{C\log(n)}.$$

We can therefore choose C sufficiently large so that

$$\mathbb{P}\left(\sup_{m\leq nT} d(X_m, \tilde{X}_m) > C\log(n)\right) \leq C_T n^{-2}$$

thus the result follows by the Borel-Cantelli lemma. $\square$

For $x \in \mathscr{T}$ recall that $|x| := d(\rho, x)$ then $|\tilde{X}_n|$ has the same distribution as a randomly trapped random walk on $\mathbb{N}$ with holding times distributed as excursion times in trees. The traps formed at different vertices are independent and identically distributed except at ρ since the root does not have an ancestor and therefore the transition probabilities from ρ differ from those at other vertices on the backbone. We now show that we can extend from $\mathbb{N}$ to $\mathbb{Z}$ with i.i.d. traps.

We begin by constructing the holding times of the randomly trapped random walk via a sequence of i.i.d. trees. Start with an initial vertex ρ and a unique ancestor $\overline{\rho}$. Attach ξ^*-1 offspring to ρ where ξ^* is size-biased as above. Note that this could result in zero offspring of ρ in which case the tree ends with only vertices ρ, $\overline{\rho}$. Otherwise, attach independent f-GW trees to the offspring of ρ. This creates a tree $\overline{\mathcal{T}}$ which has the distribution of a branch with an additional vertex connected to the root.

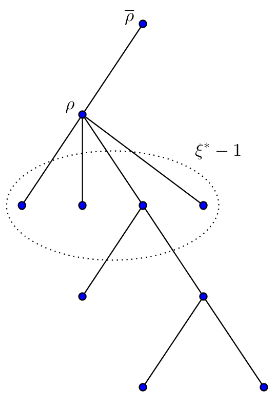

Figure 2.2: A tree $\overline{\mathcal{T}}$ with fixed vertices ρ, $\overline{\rho}$ and ξ^*-1 independent f-GW-trees attached to ρ.

Recall that $\overleftarrow{x}$ denotes the parent of $x \in \mathscr{T}$ and $c(x)$ the set of children of x.

Consider a walk $(W_n)_{n\geq 0}$ on $\overline{\mathcal{T}}$ with transition probabilities

$$P^{\overline{\mathcal{T}}}(W_{n+1}=y|W_n=x)=\begin{cases}1, & \text{if } x=y=\overline{\rho},\\ \frac{\beta+1}{\beta(|c(x)|+1)+1}, & \text{if } x=\rho,\ y=\overline{\rho},\\ \frac{\beta}{\beta(|c(x)|+1)+1}, & \text{if } x=\rho,\ y\in c(x),\\ \frac{\beta}{|c(x)|\beta+1}, & \text{if } x\notin\{\rho,\overline{\rho}\},\ y\in c(x),\\ \frac{1}{|c(x)|\beta+1}, & \text{if } x\notin\{\rho,\overline{\rho}\},\ y=\overleftarrow{x},\\ 0, & \text{otherwise.}\end{cases}$$

An excursion in $\overline{\mathcal{T}}$ started from ρ until absorption in $\overline{\rho}$ has the same distribution as the time taken to move between backbone vertices of $\mathcal{T}$ (except at the root of $\mathcal{T}$). Let $\omega=(\overline{\mathcal{T}}_x)_{x\in\mathbb{Z}}$ denote a sequence of independent trees with this law. For ω fixed let $(\eta_{x,i})_{x\in\mathbb{Z},i\geq 0}$ be independent with

$$P^{\omega}\left(\eta_{x,i}=k\right)=P_{\rho}^{\overline{\mathcal{T}}_x}\left(\min\{n>0:W_n=\overline{\rho}\}=k\right)$$

where ρ, $\overline{\rho}$ are the vertices in $\overline{\mathcal{T}}_x$ corresponding with the construction.

Recall that for a discrete time process W we let

$$\mathcal{L}^W(x,n):=\sum_{k=0}^{n}\mathbf{1}_{\{W_k=x\}}$$

denote its local time. Let $S_0=0$ and for $k=1,2,...$ define $S_k:=\inf\{n>S_{k-1}:\tilde{X}_n\neq\tilde{X}_{S_{k-1}}\}$ to be the time of the k^{th} movement of $\tilde{X}$. The walk $\tilde{Y}_n:=\tilde{X}_{S_n}$ is then a β-biased walk on $\mathcal{Y}$ reflected at ρ. Moreover, for $x\in\mathbb{Z}$ and $i=1,2,...$ we can write

$$\tilde{\eta}_{x,i}:=S_{k+1}-S_k\quad\text{where}\quad k=\min\{j:L^{\tilde{Y}}(x,j)=i\}$$

to be the holding time of $\tilde{X}$ at vertex x on the i^{th} visit.

Let $\hat{Y}_n$ be a simple, β-biased random walk on $\mathbb{Z}$, then define

$$A_n:=\sum_{k=1}^{n}\mathbf{1}_{\{\hat{Y}_k,\hat{Y}_{k-1}\geq 0\}}\qquad\text{and}\qquad A_n^{-1}:=\sup\{m\geq 0:A_m\leq n\}.$$

The process $\hat{Y}_{A_n^{-1}}$ is equal in distribution to $|\tilde{Y}_n|$ therefore, without loss of generality, we may couple $\tilde{Y}$ to $\hat{Y}$ in the construction of X so that $|\tilde{Y}_n|=\hat{Y}_{A_n^{-1}}$ without changing the distribution of X.

Let

$$\hat{S}_n := \sum_{x\in\mathbb{Z}} \sum_{i=1}^{\mathcal{L}^{\hat{Y}}(x,n-1)} \hat{\eta}_{x,i} \qquad \text{where} \qquad \hat{\eta}_{x,i} = \begin{cases} \eta_{x,i}, & \text{if } x \leq 0, \\ \tilde{\eta}_{\rho_x,i}, & \text{if } x > 0. \end{cases}$$

Write $\hat{S}_n^{-1} := \inf\{k \geq 0 : \hat{S}_k > n\}$ then $\hat{X}_n := \hat{Y}_{\hat{S}_n^{-1}}$ is a randomly trapped random walk coupled to $\tilde{X}$ with trapping times equal in distribution to $(\eta_{x,i})_{x\in\mathbb{Z},i\geq 0}$. The following lemma shows that $\tilde{X}$ and $\hat{X}$ never deviate too far.

Lemma 2.5.2. *If $\mu < 1$ and $\beta > 1$ then we have that*

$$\lim_{n\to\infty} \sup_{m\leq nT} ||\tilde{X}_m| - \hat{X}_m|$$

is $\mathbb{P}$-a.s. finite.

Proof. Using that $|\tilde{X}|$, $\hat{X}$ are discrete time processes with jump size 1, by the coupling of the two process we have that

$$\sup_{m\leq nT} ||\tilde{X}_m| - \hat{X}_m| \leq \sum_{k=0}^{\infty} \mathbf{1}_{\{\tilde{Y}_k=\rho_0\}} \tilde{\eta}_{\rho_0,\mathcal{L}^{\tilde{Y}}(\rho_0,k)} + \sum_{x\leq 0} \sum_{k=0}^{\infty} \mathbf{1}_{\{\hat{Y}_k=x\}} \eta_{x,\mathcal{L}^{\hat{Y}}(x,k)} \tag{2.15}$$

which is independent of n. That is, the supremum distance between the two processes is at most the total time spent by the two processes where the holding times differ.

By transience of the embedded walk we have that only finitely many vertices in $\mathbb{Z}^-$ are visited and each of these is only visited finitely often P-a.s. Similarly, ρ_0 and 0 are $\mathbf{P}$-a.s. visited only finitely often. By construction we have that each branch is $\mathbf{P}$-a.s. finite. We then have that all of the holding times $\{\eta_{x,j} : x \leq 0, j = 1, ..., \mathcal{L}^{\tilde{Y}}(x,\infty)\}$ and $\{\tilde{\eta}_{\rho_0,j} : j = 1, ..., \mathcal{L}^{\hat{Y}}(\rho_0,\infty)\}$ are $\mathbb{P}$-a.s. finite. It follows that the right-hand side of (2.15) is $\mathbb{P}$-a.s. finite which completes the proof.

□

Chapter 3

Speed of the random walk and central limit theorems

In this chapter we investigate biased randomly trapped random walks on $\mathbb{Z}$ and apply the results to subcritical Galton-Watson trees conditioned to survive.

In Section 3.1.1 we prove two main results. The first is Theorem 3.1 which gives conditions under which the randomly trapped random walk is ballistic and determines the value of the limiting speed.

Theorem 3.1. *Suppose $\beta > 1$ and $\mathbb{E}[\eta_0] < \infty$, then X_{nt}/n converges $\mathbb{P}$-a.s. on $D_U([0,\infty),\mathbb{R})$ as $n \to \infty$ to $\nu_\beta t$ where*

$$\nu_\beta := \frac{(\beta-1)}{\mathbb{E}[\eta_0](\beta+1)}.$$

The second main result in this section is a functional central limit theorem which we prove by considering a renewal argument similar to that of [72].

Theorem 3.2. *Suppose that $\beta > 1$ and $\mathbb{E}[\eta_0^2] < \infty$ then there exists $\varsigma^2 \in (0,\infty)$ such that*

$$B_t^n := \frac{X_{nt} - nt\nu_\beta}{\varsigma\sqrt{n}}$$

converges in $\mathbb{P}$-distribution on $D_{J_1}([0,\infty),\mathbb{R})$ as $n \to \infty$ to a standard Brownian motion.

A quenched CLT for a random walk in random environment is proved in [40] by first proving a central limit theorem for the hitting times. In Section 3.1.2 we adapt this technique to prove Theorem 3.3 which is a quenched central limit theorem with the environment dependent centring

$$\mathcal{G}^\omega(t) := \nu_\beta t - \nu_\beta \sum_{k=0}^{\lfloor \nu_\beta t - 1 \rfloor} \frac{\beta+1}{\beta-1}(E^\omega[\eta_{k,0}] - \mathbb{E}[\eta_{k,0}]).$$

Theorem 3.3. *Suppose that $\beta > 1$, $\mathbb{E}[\eta_0^2] < \infty$ and for some $\varepsilon > 0$ we have that $\mathbf{E}[E^\omega[\eta_0]^{2+\varepsilon}] < \infty$, then there exists $\vartheta \in (0,\infty)$ such that for $\mathbf{P}$-a.e. ω we have that*

$$P^\omega\left(\frac{X_t - \mathcal{G}^\omega(t)}{\vartheta\sqrt{t}} \leq x\right) \to \Phi(x) := \int_{-\infty}^x \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}}\mathrm{d}u$$

uniformly in x as $t \to \infty$.

The function $\mathcal{G}^\omega(t)$ is the annealed, deterministic centring with an environment dependent correction where this correction is a sum of centred i.i.d. random variables with non-zero variance under the environment law. This shows that the correction obeys a central limit theorem under $\mathbf{P}$, thus has $\sqrt{t}$ fluctuations and is, therefore, necessary.

In Section 3.2 we apply these results to the biased random walk on the subcritical GW-tree conditioned to survive. Recall that we let ξ denote the offspring distribution of a GW-process with mean $\mu \in (0,1)$ and variance σ^2 then let $|X_n|$ denote the graph distance between the walk at time n and the root of the tree. We begin, in Theorem 3.4, by determining the value of the speed in this model.

Theorem 3.4. *Suppose $\beta\mu < 1$, $\sigma^2 < \infty$ and $\beta > 1$, then $|X_n|/n$ converges $\mathbb{P}$-a.s. as $n \to \infty$ to*

$$\nu_\beta := \frac{\mu(\beta-1)(1-\beta\mu)}{\mu(\beta+1)(1-\beta\mu) + 2\beta(\sigma^2 - \mu(1-\mu))}.$$

The speed ν_β is unimodal with respect to the bias which remains an open problem for the supercritical tree. An example of the speed plotted against the bias is shown in Figure 3.1.

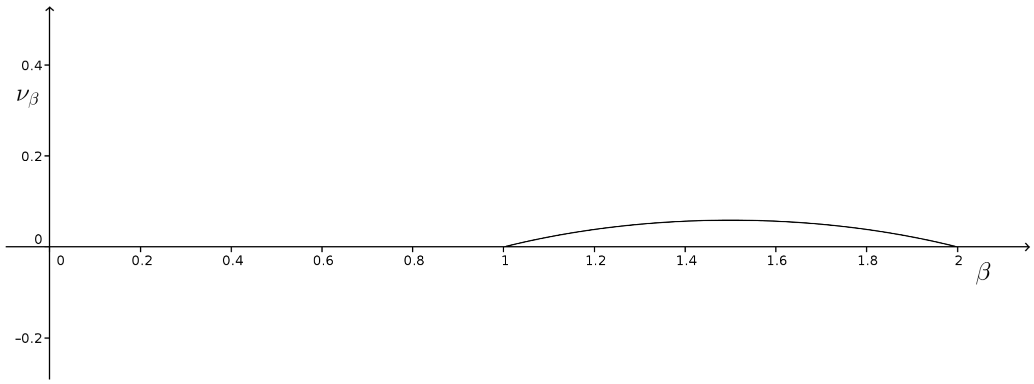

Figure 3.1: An example of the speed relative to the bias for a fixed mean $\mu = 1/2$ and variance $\sigma^2 = 1/2$.

Following on from this speed result, we give conditions such that the walk obeys an annealed functional CLT in Theorem 3.5.

Theorem 3.5. *If $\beta^2\mu < 1$ and $\mathbf{E}[\xi^3] < \infty$ then there exists $\varsigma^2 < \infty$ such that*

$$B_t^n = \frac{|X_{nt}| - nt\nu_\beta}{\varsigma\sqrt{n}}$$

converges in $\mathbb{P}$-distribution on $D_{J_1}([0,\infty),\mathbb{R})$ as $n \to \infty$ to a standard Brownian motion.

We conclude the chapter by proving the quenched analogue (Theorem 3.6) which, as for the randomly trapped random walk, requires an environment dependent centring $\mathcal{G}^{\mathcal{T}}(t)$ which we define later.

Theorem 3.6. *If $\beta^2\mu < 1$ and $\mathbf{E}[\xi^{3+\delta}] < \infty$ for some $\delta > 0$ then there exists $\vartheta > 0$ such that for $\mathbf{P}$-a.e. $\mathcal{T}$ we have that*

$$P^{\mathcal{T}}\left(\frac{|X_t| - \mathcal{G}^{\mathcal{T}}(t)}{\vartheta\sqrt{t}} \le x\right) \to \Phi(x)$$

uniformly in x as $t \to \infty$.

3.1 Randomly trapped random walks

The main aim of this section is to prove Theorems 3.1, 3.2 and 3.3 which are central limit theorems and a law of large numbers for the randomly trapped random walk.

Further to this, we will show that the speed satisfies an Einstein relation. More specifically, the derivative of the speed with respect to the bias approaches half of the diffusion coefficient of the unbiased walk as $\beta \to 1^+$.

3.1.1 A law of large numbers and functional central limit theorem

Proposition 3.1.1 is a law of large numbers for the clock process. The main ingredient is [27, Lemma 2.1] which states that the left shift on sequences $(\theta(\eta_0, \eta_1, ...) = (\eta_1, \eta_2, ...))$ acts ergodically on η under $\mathbb{P}$. This holds for any non-degenerate random walk on a fixed environment with i.i.d. traps which is why we can extend the following result to the unbiased case ($\beta = 1$). We will use this when we prove an Einstein relation for the walk.

Proposition 3.1.1. *Suppose that $\beta \ge 1$ and $\mathbb{E}[\eta_0] < \infty$ then S_{nt}/n and S_{nt}^{-1}/n converge $\mathbb{P}$-a.s. on $D_U([0,\infty),\mathbb{R})$ as $n \to \infty$ to the deterministic processes $\mathcal{S}_t = \mathbb{E}[\eta_0]t$ and $\mathcal{S}_t^{-1} = \mathbb{E}[\eta_0]^{-1}t$ respectively.*

Proof. We begin by showing convergence of S_{nt}/n. Since $\mathbb{E}[\eta_0] < \infty$ we have that $f(\eta) := \eta_0$ is integrable. Therefore, since θ acts ergodically on η under $\mathbb{P}$, by Birkhoff's ergodic theorem (see Section 2.3),

$$\lim_{n\to\infty}\frac{S_{nt}}{n} = \lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{\lfloor nt\rfloor-1}\eta_k = \lim_{n\to\infty} t\frac{1}{nt}\sum_{k=0}^{\lfloor nt\rfloor-1} f(\theta^k\eta) = t\mathbb{E}[f(\eta)]$$

almost surely. The sequence of functions S_{nt}/n are increasing in t and the limit $\mathcal{S}_t = \mathbb{E}[\eta_0]t$ is continuous therefore the convergence holds uniformly over $t \in [0, T]$ for $T < \infty$.

Since $\mathcal{S}_t$ is strictly increasing we have the desired convergence of S_{nt}^{-1}/n by *continuity of the inverse at strictly increasing functions* (Proposition 2.1.1.v). □

We can now conclude Theorem 3.1 from Proposition 3.1.1 and the results of [76] which we stated in Section 2.1.

Proof of Theorem 3.1. By definition we have that

$$\frac{X_{nt}}{n} = \frac{Y_{S^{-1}_{\lfloor nt\rfloor}}}{n}.$$

By the law of large numbers $n^{-1}Y_n$ converges $\mathbb{P}$-a.s. to $(\beta-1)/(\beta+1)$ therefore by *continuity of composition at continuous limits* (Proposition 2.1.1.i) we have the desired result. □

An additional result that can be deduced from Proposition 3.1.1 and Theorem 3.1 is that the following Einstein relation holds.

Corollary 3.1.2. *Suppose $\mathbb{E}[\eta_0] < \infty$. The unbiased ($\beta = 1$) walk $X_{\lfloor nt\rfloor}n^{-1/2}$ converges in $\mathbb{P}$-distribution on $D_{J_1}([0,\infty),\mathbb{R})$ as $n \to \infty$ to a scaled Brownian motion with variance $\Upsilon = \mathbb{E}[\eta_0]^{-1}$. Moreover,*

$$\lim_{\beta\to 1^+}\frac{\nu_\beta}{\beta-1} = \frac{\Upsilon}{2}$$

where ν_β is the speed calculated in Theorem 3.1 for the β-biased walk.

Proof. For $\beta = 1$ we have that Y_{nt} is the sum of i.i.d. copies of the random variable χ satisfying $P(\chi = 1) = 1/2 = P(\chi = -1)$ thus, by Donsker's invariance principle (see Section 2.3), $Y_{nt}n^{-1/2}$ converges in $\mathbb{P}$-distribution on $D_{J_1}([0,\infty),\mathbb{R})$ to a standard Brownian motion.

By Proposition 3.1.1 we have that $S^{-1}_{\lfloor nt\rfloor}/n$ converges $\mathbb{P}$-a.s. to the deterministic process $t/\mathbb{E}[\eta_0]$ uniformly over $t \le T$. By continuity of the limiting Brownian motion and *continuity of composition at continuous limits* (Proposition 2.1.1.i), we have that

$$\frac{X_{nt}}{\sqrt{n}} = \frac{Y_{S^{-1}_{\lfloor nt\rfloor}}}{\sqrt{n}} \qquad \text{and} \qquad \frac{Y_{nt/\mathbb{E}[\eta_0]}}{\sqrt{n}}$$

converge to the same limiting process, which is a scaled Brownian motion with variance $\Upsilon = \mathbb{E}[\eta_0]^{-1}$.

Moreover, by Theorem 3.1 we have that, for $\beta > 0$,

$$\nu_\beta = \frac{(\beta - 1)}{\mathbb{E}[\eta_0](\beta + 1)},$$

and therefore we indeed have that

$$\lim_{\beta \to 1^+} \frac{\nu_\beta}{\beta - 1} = \frac{\Upsilon}{2}.$$

□

We now move on to proving an annealed functional CLT which is the main result of this section. That is, we show that

$$B_t^n := \frac{X_{nt} - nt\nu_\beta}{\varsigma\sqrt{n}}$$

converges in $\mathbb{P}$-distribution to a standard Brownian motion for some $\varsigma^2 \in (0, \infty)$. We want to approximate X_{nt} by a sum of i.i.d. centred random variables with finite second moments. Let $\zeta_0^Y = 0$ and, for $j = 1, 2, ...,$ define $\zeta_j^Y := \inf\{m > \zeta_{j-1}^Y : \{Y_l\}_{l=0}^{m-1} \cap \{Y_l\}_{l=m}^{\infty} = \emptyset\}$ to be the regeneration times of the walk Y. We then have that $\zeta_j^X := S_{\zeta_j^Y}$ for $j \geq 1$ are regeneration times for X, $\varrho_j := Y_{\zeta_j^Y} = X_{\zeta_j^X}$ are the regeneration points and we write

$$\chi_j := \left(X_{\zeta_j^X} - X_{\zeta_{j-1}^X} - \left(\zeta_j^X - \zeta_{j-1}^X\right)\nu_\beta\right) = \left(\varrho_j - \varrho_{j-1} - \left(\zeta_j^X - \zeta_{j-1}^X\right)\nu_\beta\right).$$

By Lemma 2.3.7 the time and distance between regenerations of Y have exponential moments, that is

$$\mathbb{P}(\varrho_{j+1} - \varrho_j > n),\ \mathbb{P}(\zeta_{j+1}^Y - \zeta_j^Y > n) \leq Ce^{-cn} \tag{3.1}$$

for any $j \geq 1$ and some constants C, c.

Lemma 3.1.3. *Suppose that $\beta > 1$ and $\mathbb{E}[\eta_0] < \infty$ then $\{\chi_j\}_{j\geq 2}$ are centred and i.i.d. under $\mathbb{P}$.*

Proof. By [30] we have that the sections of the walk $(Y_{i+\zeta_j^Y} - \varrho_j)_{i=0}^{\zeta_{j+1}^Y - \zeta_j^Y - 1}$, for $j \geq 1$, are i.i.d. therefore, since the traps $(\omega_x)_{x\in\mathbb{Z}}$ are i.i.d. and independent of the embedded walk Y, we have that the sequences $(\eta_k)_{k=\zeta_j^Y}^{\zeta_{j+1}^Y - 1}$ are i.i.d. It follows that $\{\chi_j\}_{j\geq 2}$ are i.i.d. under $\mathbb{P}$.

It remains to show that χ_j are centred. Since the distribution of a given holding time is independent of the regeneration times of Y and $\mathbb{E}[\eta_0] < \infty$ we have that

$$\nu_\beta \mathbb{E}[\zeta_2^X - \zeta_1^X] = \frac{\beta - 1}{(\beta+1)\mathbb{E}[\eta_0]} \mathbb{E}\left[\sum_{k=\zeta_1^Y}^{\zeta_2^Y - 1} \mathbb{E}[\eta_k | \zeta_2^Y, \zeta_1^Y]\right] = \frac{\beta - 1}{\beta + 1} \mathbb{E}[\zeta_2^Y - \zeta_1^Y]. \tag{3.2}$$

We want to show this is equal to $\mathbb{E}[\varrho_j - \varrho_{j-1}] = \mathbb{E}[Y_{\zeta_2^Y} - Y_{\zeta_1^Y}]$. By (3.1) the time between regenerations and distance between regeneration points have exponential moments hence, by the law of large numbers,

$$\frac{\sum_{j=2}^m \varrho_j - \varrho_{j-1}}{m} \to \mathbb{E}[\varrho_2 - \varrho_1],$$
$$\frac{\sum_{j=2}^m \zeta_j^Y - \zeta_{j-1}^Y}{m} \to \mathbb{E}[\zeta_2^Y - \zeta_1^Y]$$

$\mathbb{P}$-a.s. and, therefore,

$$\frac{\sum_{j=2}^m \varrho_j - \varrho_{j-1}}{\sum_{j=2}^m \zeta_j^Y - \zeta_{j-1}^Y} \to \frac{\mathbb{E}[\varrho_2 - \varrho_1]}{\mathbb{E}[\zeta_2^Y - \zeta_1^Y]} \tag{3.3}$$

$\mathbb{P}$-a.s. as $m \to \infty$. However,

$$\frac{\sum_{j=2}^m \varrho_j - \varrho_{j-1}}{\sum_{j=2}^m \zeta_j^Y - \zeta_{j-1}^Y} = \frac{Y_{\zeta_m^Y}}{\zeta_m^Y}\left(1 + \frac{\zeta_1^Y}{\zeta_m^Y - \zeta_1^Y}\right) - \frac{\varrho_1}{\zeta_m^Y - \zeta_1^Y}$$

where $\varrho_1/(\zeta_m^Y - \zeta_1^Y)$ and $\zeta_1^Y/(\zeta_m^Y - \zeta_1^Y)$ converge $\mathbb{P}$-a.s. to 0. Furthermore, by the law of large numbers, $Y_{\zeta_m^Y}/\zeta_m^Y$ converges $\mathbb{P}$-a.s. to $(\beta - 1)/(\beta + 1)$ therefore

$$\frac{\sum_{j=2}^m \varrho_j - \varrho_{j-1}}{\sum_{j=2}^m \zeta_j^Y - \zeta_{j-1}^Y} \to \frac{\beta - 1}{\beta + 1} \tag{3.4}$$

$\mathbb{P}$-a.s. By (3.2), (3.3) and (3.4) we then have that

$$\mathbb{E}[\varrho_2 - \varrho_1] = \frac{\beta - 1}{\beta + 1} \mathbb{E}[\zeta_2^Y - \zeta_1^Y] = \nu_\beta \mathbb{E}[\zeta_2^X - \zeta_1^X].$$

Therefore χ_j are centred as desired. □

In Theorem 3.2 we show that B_t^n can be approximated by a sum of χ_j which, by Lemma 3.1.3, are i.i.d. centred random variables. With the aim of proving a central limit theorem, we now show that they also have finite second moments.

Lemma 3.1.4. *Suppose that $\beta > 1$ and $\mathbb{E}[\eta_0^2] < \infty$ then $\mathbb{E}[\chi_j^2] < \infty$ for $j \geq 2$.*

Proof. Since $\{\chi_j\}_{j \geq 2}$ are i.i.d. under $\mathbb{P}$ we have that $\mathrm{Var}_{\mathbb{P}}(\chi_j) = \mathrm{Var}_{\mathbb{P}}(\chi_2)$ for all $j \geq 2$. By properties of regenerations times $\varrho_2 \geq \varrho_1$ and $\zeta_2^X \geq \zeta_1^X$ almost surely therefore we have that

$$\mathrm{Var}_{\mathbb{P}}(\chi_2) \leq \mathbb{E}\left[(\varrho_2 - \varrho_1)^2\right] + \nu_\beta^2 \mathbb{E}\left[(\zeta_2^X - \zeta_1^X)^2\right]. \tag{3.5}$$

For the second term we have

$$\begin{aligned}
&\mathbb{E}\left[(\zeta_2^X - \zeta_1^X)^2\right] \\
&= \mathbb{E}\left[\left(\sum_{x=\varrho_1}^{\varrho_2-1} \sum_{i=1}^{\mathcal{L}(x,\infty)} \eta_{x,i}\right)^2\right] \\
&= \mathbb{E}\left[\sum_{x=\varrho_1}^{\varrho_2-1}\left(\sum_{i=1}^{\mathcal{L}(x,\infty)} \eta_{x,i}\right)^2\right] + \mathbb{E}\left[\sum_{x=\varrho_1}^{\varrho_2-1}\sum_{y=\varrho_1}^{\varrho_2-1}\mathbf{1}_{\{x\neq y\}}\left(\sum_{i=1}^{\mathcal{L}(x,\infty)} \eta_{x,i}\right)\left(\sum_{j=1}^{\mathcal{L}(y,\infty)} \eta_{y,j}\right)\right].
\end{aligned} \tag{3.6}$$

By conditioning on Y we have that the holding times at separate vertices are independent therefore the second term in this expression can be written as

$$\begin{aligned}
&\mathbb{E}\left[\sum_{x=\varrho_1}^{\varrho_2-1}\sum_{y=\varrho_1}^{\varrho_2-1}\mathbf{1}_{\{x\neq y\}}\mathbb{E}\left[\sum_{i=1}^{\mathcal{L}(x,\infty)} \eta_{x,i}\middle|Y\right]\mathbb{E}\left[\sum_{j=1}^{\mathcal{L}(y,\infty)} \eta_{y,j}\middle|Y\right]\right] \\
&\qquad = \mathbb{E}\left[\sum_{x=\varrho_1}^{\varrho_2-1}\sum_{y=\varrho_1}^{\varrho_2-1}\mathbf{1}_{\{x\neq y\}}\left(\sum_{i=1}^{\mathcal{L}(x,\infty)} \mathbb{E}[\eta_{x,i}]\right)\left(\sum_{j=1}^{\mathcal{L}(y,\infty)} \mathbb{E}[\eta_{y,j}]\right)\right] \\
&\qquad = \mathbb{E}\left[\sum_{x=\varrho_1}^{\varrho_2-1}\sum_{y=\varrho_1}^{\varrho_2-1}\mathbf{1}_{\{x\neq y\}}\mathbb{E}[\eta_0]^2\mathcal{L}(x,\infty)\mathcal{L}(y,\infty)\right] \\
&\qquad \leq \mathbb{E}[\eta_0]^2\mathbb{E}\left[\left(\sum_{x=\varrho_1}^{\varrho_2-1}\mathcal{L}(x,\infty)\right)^2\right] \\
&\qquad = \mathbb{E}[\eta_0]^2\mathbb{E}[(\zeta_2^Y - \zeta_1^Y)^2].
\end{aligned}$$

By (3.1), the time between regenerations $\zeta_2^Y - \zeta_1^Y$ has exponential moments therefore $\mathbb{E}[(\zeta_2^Y - \zeta_1^Y)^2] < \infty$. Furthermore, since Y moves in discrete time and has jumps of length 1

$$\mathbb{E}\left[(\varrho_2 - \varrho_1)^2\right] \leq \mathbb{E}\left[(\zeta_2^Y - \zeta_1^Y)^2\right] < \infty. \tag{3.7}$$

Combining (3.7) with (3.5) and (3.6), in order to show that $\mathrm{Var}_{\mathbb{P}}(\chi_2) < \infty$ it remains to show that

$$\mathbb{E}\left[\sum_{x=\varrho_1}^{\varrho_2-1}\left(\sum_{i=1}^{\mathcal{L}(x,\infty)} \eta_{x,i}\right)^2\right] < \infty.$$

Conditioning on Y this expectation is equal to

$$\mathbb{E}\left[\sum_{x=\varrho_1}^{\varrho_2-1}\sum_{i,j=1}^{\mathcal{L}(x,\infty)}\mathbb{E}\left[\eta_{x,i}\eta_{x,j}\middle|Y\right]\right] \;\leq\; \mathbb{E}\left[\sum_{x=\varrho_1}^{\varrho_2-1}\mathcal{L}(x,\infty)^2\mathbb{E}\left[\eta_{x,1}^2\right]\right] \;\leq\; \mathbb{E}[\eta_0^2]\mathbb{E}[(\zeta_2^Y-\zeta_1^Y)^2]$$

which is finite by the assumptions of the theorem and equation (3.7). $\square$

We now conclude the proof of the annealed functional central limit theorem by showing that B_t^n can be suitably approximated by a sum of χ_j.

Proof of Theorem 3.2. By Lemmas 3.1.3 and 3.1.4

$$\Sigma_m := \sum_{j=2}^{m}\chi_j = \left(X_{\zeta_m^X}-\zeta_m^X\nu_\beta\right)-\left(\varrho_1-\zeta_1^X\nu_\beta\right)$$

for $m\geq 2$ is a sum of i.i.d. centred random variables with finite second moment.

Write $m_t := \sup\{j\geq 0 : \zeta_j^X\leq t\}$ to be the number of regenerations by time $t>0$ then

$$\sup_{t\in[0,T]}\left|B_t^n-\frac{\Sigma_{m_{tn}}}{\varsigma\sqrt{n}}\right| \leq \frac{\varrho_1+\zeta_1^X+|\min_k Y_k|}{\varsigma\sqrt{n}}+\sup_{j=1,\dots,m_{Tn}}\frac{\varrho_{j+1}-\varrho_j+\left(\zeta_{j+1}^X-\zeta_j^X\right)\nu_\beta}{\varsigma\sqrt{n}}.$$

The random variables ϱ_1, ζ_1^X and $|\min_k Y_k|$ are all almost surely finite therefore the first fraction converges to 0 $\mathbb{P}$-a.s. For $\varepsilon>0$, by a union bound and Markov's inequality

$$\begin{aligned}\mathbb{P}&\left(\sup_{j=1,\dots,m_{Tn}}\frac{\varrho_{j+1}-\varrho_j}{\sqrt{n}}>\varepsilon\right)\\ &\leq \mathbb{P}\left(m_{Tn}>2Tn\mathbb{E}[\eta_0]^{-1}\right)+C_Tn\mathbb{P}\left(\varrho_2-\varrho_1>\varepsilon\sqrt{n}\right)\\ &\leq \mathbb{P}\left(m_{Tn}>2Tn\mathbb{E}[\eta_0]^{-1}\right)+C_{T,\varepsilon}\mathbb{E}\left[(\varrho_2-\varrho_1)^2\mathbf{1}_{\{\varrho_2-\varrho_1\geq\varepsilon\sqrt{n}\}}\right].\end{aligned}$$

By Proposition 3.1.1, since $S_t^{-1}\geq m_t$, we have that $\mathbb{P}\left(m_{Tn}>2Tn\mathbb{E}[\eta_0]^{-1}\right)\to 0$ as $n\to\infty$. By (3.7) we have that $\mathbb{E}\left[(\varrho_2-\varrho_1)^2\right]<\infty$ therefore by dominated convergence

$$\mathbb{E}\left[(\varrho_2-\varrho_1)^2\mathbf{1}_{\{\varrho_2-\varrho_1\geq\varepsilon\sqrt{n}\}}\right]\to 0$$

as $n\to\infty$. Similarly, by Lemma 3.1.4, $\mathbb{E}\left[(\zeta_2^X-\zeta_1^X)^2\right]<\infty$ hence we have that

$$\mathbb{P}\left(\sup_{j=1,\dots,m_{Tn}}\frac{\zeta_{j+1}^X-\zeta_j^X}{\sqrt{n}}>\varepsilon\right)\to 0$$

as $n\to\infty$, and the supremum distance between $(B_t^n)_{t\in[0,T]}$ and $(\Sigma_{m_{tn}}/\varsigma\sqrt{n})_{t\in[0,T]}$ converges to 0 in $\mathbb{P}$-probability. It therefore suffices to prove an invariance principle for $\Sigma_{m_{tn}}$.

For $s \in [0,\infty)$ let Σ_s denote the linear interpolation of Σ_m then by Donsker's invariance principle (see Section 2.3) we have that $(\Sigma_{tn}/\sqrt{n})_{t\in[0,T]}$ converges in distribution to a scaled Brownian motion.

By the law of large numbers we have that ζ_n^Y/n converges $\mathbb{P}$-a.s. to $\mathbb{E}[\zeta_2^Y - \zeta_1^Y]$ as $n \to \infty$. Therefore, by *continuity of the inverse at strictly increasing functions* (Proposition 2.1.1.v), we have that m_{tn}/n converges $\mathbb{P}$-a.s. on $D_{J_1}([0,\infty),\mathbb{R})$ to the deterministic process $R_t := (\mathbb{E}[\eta_0]\mathbb{E}[\zeta_2^Y - \zeta_1^Y])^{-1}t$.

By *continuity of composition at continuous limits* (Proposition 2.1.1.i) the sequence $(\Sigma_{m_{tn}}/\sqrt{n})_{t\in[0,T]}$ converges to the same limit as $(\Sigma_{R_t n}/\sqrt{n})_{t\in[0,T]}$ which is a scaled Brownian motion. In particular, choosing

$$\varsigma^2 = \frac{\mathbb{E}[\chi_2^2]}{\mathbb{E}[\eta_0]\mathbb{E}[\zeta_2^Y - \zeta_1^Y]} \tag{3.8}$$

we have that B_t^n converges to a standard Brownian motion. $\square$

3.1.2 A quenched central limit theorem with environment dependent centring

In this section we prove a quenched central limit theorem for the randomly trapped random walk under a $2+\varepsilon$ moment condition. We do this by adapting the method used in [40] and first proving a quenched CLT for the first hitting time of n.

Write $\tau_n := \inf\{t \geq 0 : X_t = n\}$ and, for ω fixed, $\mathcal{H}^\omega(n) := E^\omega[\tau_n]$. Let $\upsilon_k := \tau_{k+1} - \tau_k$ be the time taken between hitting k and $k+1$ for the first time then the elements of $(\upsilon_k)_{k\geq 1}$ are independent under P^ω and

$$\tau_n = \sum_{k=0}^{n-1} \upsilon_k.$$

Lemma 3.1.5. *Suppose that $\beta > 1$ and $\mathbb{E}[\eta_0^2] < \infty$, then for $\mathbf{P}$-a.e. ω we have that*

$$P^\omega\left(\frac{\tau_n - \mathcal{H}^\omega(n)}{\sigma\sqrt{n}} < x\right) \to \Phi(x)$$

uniformly in x as $n \to \infty$, where $\sigma^2 = \mathbf{E}[\mathrm{Var}_\omega(\tau_1)]$.

Proof. By definition of τ_n, $\mathcal{H}^\omega(n)$ and υ_k

$$\frac{\tau_n - \mathcal{H}^\omega(n)}{\sigma\sqrt{n}} = \frac{\sum_{k=0}^{n-1}(\upsilon_k - E^\omega[\upsilon_k])}{\sigma\sqrt{n}}.$$

It therefore suffices to show that Lindeberg's conditions (see [33, Theorem 3.4.5]) hold:

1. for $\mathbf{P}$-a.e. ω, as $n \to \infty$

$$\sum_{k=0}^{n-1} E^{\omega}\left[\left(\frac{\upsilon_k - E^{\omega}[\upsilon_k]}{\sigma\sqrt{n}}\right)^2\right] \to 1;$$

2. for $\mathbf{P}$-a.e. ω, $\forall\varepsilon > 0$ as $n \to \infty$

$$\sum_{k=0}^{n-1} E^{\omega}\left[\left(\frac{\upsilon_k - E^{\omega}[\upsilon_k]}{\sigma\sqrt{n}}\right)^2 \mathbf{1}_{\{|\upsilon_k - E^{\omega}[\upsilon_k]| > \varepsilon\sqrt{n}\}}\right] \to 0.$$

Recall, from the remark prior to Proposition 3.1.1, that θ is the shift map which is ergodic by [27, Lemma 2.1]. For the first condition we have that $\upsilon_k - E^{\omega}[\upsilon_k] = \theta^k(\upsilon_0 - E^{\omega}[\upsilon_0])$. These random variables are identically distributed under $\mathbf{P}$ with $\mathbf{E}\left[E^{\omega}[(\upsilon_0 - E^{\omega}[\upsilon_0])^2]\right] < \infty$ therefore

$$\sum_{k=0}^{n-1} \frac{\mathrm{Var}_{\omega}(\upsilon_k)}{\sigma^2 n} = \sum_{k=0}^{n-1} \frac{\mathrm{Var}_{\theta^k\omega}(\upsilon_0)}{\sigma^2 n}$$

which converges to $\mathbf{E}\left[\mathrm{Var}_{\omega}(\upsilon_0)\right]\sigma^{-2} = 1$ for $\mathbf{P}$-a.e. ω by Birkhoff's ergodic theorem (see Section 2.3).

For the second condition write $U_K^{\omega}(\cdot) := E^{\omega}[(\cdot - E^{\omega}[\cdot])^2 \mathbf{1}_{\{|\cdot - E^{\omega}[\cdot]| > K\}}]$ then for all $\varepsilon > 0$ there exists $\exists N_{\varepsilon,K} \in \mathbb{N}$ such that $\varepsilon\sqrt{n} > K$ for all $n \geq N_{\varepsilon,K}$. Therefore, for n large

$$\sum_{k=0}^{n-1} E^{\omega}\left[\left(\frac{\upsilon_k - E^{\omega}[\upsilon_k]}{\sigma\sqrt{n}}\right)^2 \mathbf{1}_{\{|\upsilon_k - E^{\omega}[\upsilon_k]| > \varepsilon\sqrt{n}\}}\right] \leq \sum_{k=0}^{n-1} \frac{U_K^{\omega}(\upsilon_k)}{\sigma^2 n} = \sum_{k=0}^{n-1} \frac{U_K^{\theta^k\omega}(\upsilon_0)}{\sigma^2 n}.$$

By Birkhoff's ergodic theorem, for $\mathbf{P}$-a.e. ω, this converges to

$$\frac{\mathbf{E}\left[U_K^{\omega}(\upsilon_0)\right]}{\sigma^2} = \frac{\mathbf{E}\left[E^{\omega}\left[(\upsilon_0 - E^{\omega}[\upsilon_0])^2 \mathbf{1}_{\{|\upsilon_0 - E^{\omega}[\upsilon_0]| > K\}}\right]\right]}{\sigma^2}$$

which converges to 0 as $K \to \infty$ by dominated convergence. □

By Lemma 3.1.5 we have a central limit theorem for the first hitting time of vertex n. The environment dependent centring $\mathcal{H}^{\omega}(n)$ can be written as the sum of n identically distributed random variables $E^{\omega}[\upsilon_k]$. These are not independent however; they are only locally dependent. Recall that $\eta_{k,i}$ is the i^{th} holding time at vertex k hence $\mathbb{E}[\upsilon_0] = \mathbb{E}[\eta_{0,0}](\beta+1)/(\beta-1)$ then write

$$\tilde{\mathcal{H}}^{\omega}(n) := \sum_{k=0}^{n-1} \frac{\beta+1}{\beta-1} E^{\omega}[\eta_{k,0}].$$

We now show that $\mathcal{H}^\omega$ and $\tilde{\mathcal{H}}^\omega$ do not differ too much and therefore Lemma 3.1.5 also holds with $\mathcal{H}^\omega$ replaced by $\tilde{\mathcal{H}}^\omega$. Under $\mathbf{P}$, the function $\tilde{\mathcal{H}}^\omega(n)$ is a sum of i.i.d. random variables with non-zero variance unless $E^\omega[\eta_0]$ is constant. We thus have a central limit theorem for $\tilde{\mathcal{H}}^\omega$ (and therefore $\mathcal{H}^\omega$), which will show that the environment dependent centring is necessary. Notice that this is the first point at which we introduce the extra $2+\varepsilon$ moment condition however we do require the condition later in Lemma 3.1.7 as well.

Lemma 3.1.6. *Suppose $\beta > 1$ and that $\mathbf{E}\left[E^\omega[\eta_0]^{2+\varepsilon}\right] < \infty$ for some $\varepsilon > 0$, then for $\mathbf{P}$-a.e. ω*

$$\left|\frac{\tilde{\mathcal{H}}^\omega(n) - \mathcal{H}^\omega(n)}{\sqrt{n}}\right| \to 0.$$

Proof. Let $\tau_n^Y := \inf\{m \geq 0 : Y_m = n\}$ be the hitting time of level n by the embedded walk. Recall that $\mathcal{L}(k,m)$ denotes the local time of Y at vertex k by time m and that the trapping times $\eta_{k,j}$ do not depend on the embedded walk, then

$$\mathcal{H}^\omega(n) \;=\; E^\omega\left[\sum_{k=-\infty}^{n-1}\sum_{j=1}^{\mathcal{L}(k,\tau_n^Y)}\eta_{k,j}\right] \;=\; \sum_{k=-\infty}^{n-1} E_0[\mathcal{L}(k,\tau_n^Y)]E^\omega[\eta_{k,0}].$$

We need to determine the expected local times at sites up to reaching level n. By the strong Markov property we have that $E_0[\mathcal{L}(k,\tau_n^Y)] = P_0(\tau_k^Y < \tau_n^Y)E_k[\mathcal{L}(k,\tau_n^Y)]$.

Let $(\tau_n^Y)^+ := \inf\{m > 0 : Y_m = n\}$ be the first return time to level n by the embedded walk. By Lemma 2.3.2, the number of visits to k before reaching n for a walk started at $k < n$ is geometrically distributed with escape probability

$$P_k(\tau_n^Y < (\tau_k^Y)^+) = \frac{\beta}{1+\beta}\left(1 - P_0(\tau_{-1}^Y < \tau_{n-k-1}^Y)\right) = \frac{\beta^{n-k}(\beta-1)}{(\beta^{n-k}-1)(\beta+1)}.$$

Therefore,

$$E_k[\mathcal{L}(k,\tau_n^Y)] = \frac{(\beta^{n-k}-1)(\beta+1)}{\beta^{n-k}(\beta-1)}$$

and

$$E_0[\mathcal{L}(k,\tau_n^Y)] = \begin{cases} \frac{(\beta^n-1)(\beta+1)}{\beta^{n-k}(\beta-1)} & \text{if } k < 0 \\ \frac{(\beta^{n-k}-1)(\beta+1)}{\beta^{n-k}(\beta-1)} & \text{if } k \geq 0. \end{cases}$$

For fixed $k < 0$ we have that $E_0[\mathcal{L}(k,\tau_n^Y)]$ is increasing in n and converges as $n \to \infty$ to $\beta^k(\beta+1)/(\beta-1)$. In particular,

$$0 \leq \sum_{k=-\infty}^{-1} E_0[\mathcal{L}(k,\tau_n^Y)]E^\omega[\eta_{k,0}] \leq C\sum_{k=1}^{\infty}\beta^{-k}E^\omega[\eta_{-k,0}]$$

which is finite for $\mathbf{P}$-a.e. ω therefore $n^{-1/2}\sum_{k=-\infty}^{-1} E_0[\mathcal{L}(k,\tau_n^Y)]E^\omega[\eta_{k,0}]$ converges to

0 for $\mathbf{P}$-a.e. ω.

For $k \geq 0$ fixed, $E_0[\mathcal{L}(k, \tau_n^Y)]$ is increasing in n and converges to $(\beta+1)/(\beta-1)$. In particular,

$$\begin{aligned}
0 &\leq \sum_{k=0}^{\infty} \left(\frac{\beta+1}{\beta-1} - E_0[\mathcal{L}(k, \tau_n^Y)] \right) E^\omega[\eta_{k,0}] \\
&= \frac{\beta+1}{\beta-1} \sum_{k=0}^{n-1} \beta^{-(n-k)} E^\omega[\eta_{k,0}] \\
&= \frac{\beta+1}{\beta-1} \left(\sum_{k=0}^{n-\left\lfloor 2\frac{\log(n)}{\log(\beta)} \right\rfloor} \beta^{-(n-k)} E^\omega[\eta_{k,0}] + \sum_{k=n-\left\lfloor 2\frac{\log(n)}{\log(\beta)} \right\rfloor+1}^{n-1} \beta^{-(n-k)} E^\omega[\eta_{k,0}] \right) \\
&\leq C \left(\sum_{k=0}^{n-1} \beta^{-2\frac{\log(n)}{\log(\beta)}} E^\omega[\eta_{k,0}] + \sum_{k=n-\left\lfloor 2\frac{\log(n)}{\log(\beta)} \right\rfloor+1}^{n-1} E^\omega[\eta_{k,0}] \right) \\
&= C \left(\sum_{k=0}^{n-1} \frac{E^\omega[\eta_{k,0}]}{n^2} + \sum_{k=n-\left\lfloor 2\frac{\log(n)}{\log(\beta)} \right\rfloor+1}^{n-1} E^\omega[\eta_{k,0}] \right).
\end{aligned}$$

The first term converges to 0 for $\mathbf{P}$-a.e. ω by the strong law of large numbers. For the second term we have that, for $\delta, \epsilon > 0$, by Markov's inequality

$$\begin{aligned}
\mathbf{P}\left(\sum_{k=n-\left\lfloor 2\frac{\log(n)}{\log(\beta)} \right\rfloor+1}^{n-1} E^\omega[\eta_{k,0}] > \epsilon\sqrt{n} \right) &\leq 2\frac{\log(n)}{\log(\beta)} \mathbf{P}\left(E^\omega[\eta_0] > \frac{\epsilon\sqrt{n}\log(\beta)}{2\log(n)} \right) \\
&= 2\frac{\log(n)}{\log(\beta)} \mathbf{P}\left(E^\omega[\eta_0]^{2+\delta} > \frac{C_\epsilon n^{1+\delta/2}}{\log(n)^{2+\delta}} \right) \\
&\leq \frac{C\log(n)^{3+\delta}}{n^{1+\delta/2}}
\end{aligned}$$

since we can choose $\delta > 0$ sufficiently small such that $\mathbf{E}\left[E^\omega[\eta_0]^{2+\delta}\right] < \infty$. By the Borel-Cantelli lemma we then have that

$$\sum_{k=n-\left\lfloor 2\frac{\log(n)}{\log(\beta)} \right\rfloor+1}^{n-1} \frac{E^\omega[\eta_{k,0}]}{\sqrt{n}}$$

converges to 0 for $\mathbf{P}$-a.e. ω. □

We now prove a technical lemma that allows us to control the difference between $\tilde{\mathcal{H}}^\omega$ and its expected value under $\mathbf{P}$ which is important in proving the quenched CLT for the walk.

Lemma 3.1.7. *Let*

$$\mathcal{J}(n) := \sum_{j=0}^{n-1} \left(E^\omega[\eta_{k,0}] - \mathbb{E}[\eta_{k,0}]\right)$$

and

$$\mathcal{J}^*(n) := \max_{m\le n} \mathcal{J}(m).$$

1. *Suppose* $\mathbf{E}\left[E^\omega[\eta_0]^2\right] < \infty$*, then for any* $c > 0$*,* $\mathcal{J}(n)n^{-\frac{1+c}{2}} \to 0$ *for* $\mathbf{P}$*-a.e.* ω*;*
2. *Suppose* $\mathbf{E}\left[E^\omega[\eta_0]^{2+\varepsilon}\right] < \infty$ *for some* $\varepsilon > 0$*, then for* $\delta > 0$ *sufficiently small and some constant* C

$$\mathbf{E}\left[|\mathcal{J}^*(n)|^{2+2\delta}\right]^{\frac{1}{2+2\delta}} \le Cn^{1/2}.$$

Proof. By [66, Theorem IX.3.17], if Z_n are i.i.d. centred random variables, a_n is an increasing, diverging sequence and

a) $\sum_{n=1}^\infty \mathbf{P}(|Z_1| \ge a_n) < \infty$;

b) $\sum_{k=n}^\infty a_k^{-2} = O\left(\frac{n}{a_n^2}\right)$;

c) $a_k/a_n \le Ck/n$ for all $k \ge n$

then $\sum_{k=1}^n Z_k/a_n$ converges to 0, $\mathbf{P}$-a.s.

Write $Z_n := E^\omega[\eta_{n,0}] - \mathbb{E}[\eta_{n,0}]$ then Z_n are i.i.d. and centred under $\mathbf{P}$; moreover, the sequence $a_n = n^{\frac{1+c}{2}}$ is increasing and diverges. By Chebyshev's inequality we have that

$$\sum_{n=1}^\infty \mathbf{P}(|Z_1| \ge a_n) \le \sum_{n=1}^\infty \mathrm{Var}_{\mathbf{P}}(E^\omega[\eta_0])n^{-(1+c)} < \infty$$

which gives (a). Since $n/a_n^2 = n^{-c}$, an integral test gives (b). For $k \ge n$ we have that $a_k/a_n = (k/n)^{\frac{1+c}{2}} \le k/n$ so long as $c \le 1$ which gives (c). We therefore have that for any $c > 0$, $\mathcal{J}(n)n^{-\frac{1+c}{2}} \to 0$ for $\mathbf{P}$-a.e. ω hence statement 1 holds.

The process $\mathcal{J}(m)$ is a martingale therefore by the L^p-maximal inequality we have that

$$\mathbf{E}\left[\max_{m\le n} |\mathcal{J}(m)|^{2+2\delta}\right] \le \left(\frac{2+2\delta}{1+2\delta}\right)^{2+2\delta} \mathbf{E}\left[|\mathcal{J}(n)|^{2+2\delta}\right].$$

It therefore suffices to show that

$$\mathbf{E}\left[\left(\sum_{j=0}^{n-1} \frac{E^\omega[\eta_{k,0}] - \mathbb{E}[\eta_{k,0}]}{\sqrt{n}}\right)^{2+2\delta}\right]$$

is bounded above. By the Marcinkiewicz-Zygmund inequality [57, Theorem 5] we have that

$$\mathbf{E}\left[\left(\sum_{j=0}^{n-1}\frac{E^\omega[\eta_{k,0}]-\mathbb{E}[\eta_{k,0}]}{\sqrt{n}}\right)^{2+2\delta}\right] \leq C\mathbf{E}\left[\left(\sum_{j=0}^{n-1}\left(\frac{E^\omega[\eta_{k,0}]-\mathbb{E}[\eta_{k,0}]}{\sqrt{n}}\right)^2\right)^{1+\delta}\right]$$

which is bounded above by

$$C\mathbf{E}\left[\sum_{j=0}^{n-1}\frac{(E^\omega[\eta_{k,0}]-\mathbb{E}[\eta_{k,0}])^{2+2\delta}}{n}\right] = C\mathbf{E}\left[(E^\omega[\eta_0]-\mathbb{E}[\eta_0])^{2+2\delta}\right]$$

using Jensen's inequality. Using that $\mathbf{E}\left[E^\omega[\eta_0]^{2+\varepsilon}\right] < \infty$ for some $\varepsilon > 0$, it then follows that for $\delta > 0$ sufficiently small and some constant C

$$\mathbf{E}\left[|\mathcal{J}^*(n)|^{2+2\delta}\right]^{\frac{1}{2+2\delta}} \leq Cn^{1/2}.$$

□

We now prove the main result of the section which is a quenched central limit theorem for the randomly trapped random walk. Recall from Theorem 3.1 that $\nu_\beta = (\beta-1)\left((\beta+1)\mathbb{E}[\eta_0]\right)^{-1}$ is the $\mathbb{P}$-a.s. limit of X_n/n and from Lemma 3.1.5 that $\sigma^2 = \mathbf{E}[\mathrm{Var}_\omega(\tau_1)]$ is the variance in the quenched CLT for the first hitting times τ_n. Write $\vartheta := \sigma\nu_\beta^{3/2}$ and recall that

$$\mathcal{G}^\omega(t) := \nu_\beta t - \nu_\beta \sum_{k=0}^{\lfloor \nu_\beta t-1\rfloor}\frac{\beta+1}{\beta-1}(E^\omega[\eta_{k,0}]-\mathbb{E}[\eta_{k,0}]).$$

Proof of Theorem 3.3. Let $\overline{X}_t := \sup\{|X_s| : \ s \leq t\}$ be the furthest point reached by X up to time t; then $\tau_{\overline{X}_t} \leq t < \tau_{\overline{X}_t+1}$. We then have that $|X_t - \overline{X}_t| \leq \sup_{s\geq \tau_{\overline{X}_t}} \overline{X}_t - X_s$. Since X_{nt}/n converges $\mathbb{P}$-a.s. to $\nu_\beta t$ uniformly over t we can choose a constant C_T such that for n sufficiently large $X_t \leq C_T n$ for all $t \leq nT$. Write

$$A_n := \bigcap_{k=1}^{\lfloor C_T n+1\rfloor}\left\{\inf\{Y_m : m \geq \tau_k^Y\} \geq k - C\log(n)\right\}$$

to be the event that the walk never backtracks distance $C\log(n)$ up to reaching vertex $\lceil C_T n\rceil$. By Lemma 2.3.2 we then have that

$$P(A_n^c) \ \leq \ C_T n P(\tau_{-\lfloor C\log(n)\rfloor} < \infty) \ \leq \ C_T n \beta^{-C\log(n)} \ = \ C_T n^{1-C\log(\beta)}.$$

Therefore, choosing C such that $C\log(\beta) > 2$, by the Borel-Cantelli lemma we have that there exists only finitely many n such that the walk backtracks distance $C\log(n)$ up to time nT. In particular, on this event $|X_t - \overline{X}_t|n^{-1/2} \leq C\log(n)n^{-1/2}$ which converges deterministically to 0 uniformly over $t \leq T$. It therefore suffices to show that for **P**-a.e. ω

$$\lim_{t\to\infty} P^\omega\left(\frac{\overline{X}_t - \mathcal{G}^\omega(t)}{\vartheta\sqrt{t}} < x\right) = \Phi(x).$$

By monotonicity we have that $\{\overline{X}_t \leq m\} = \{\tau_{m+1} > t\}$. Writing $\mathcal{I}^\omega(t) := \lfloor x\vartheta\sqrt{t} + \mathcal{G}^\omega(t) + 1\rfloor$ it then follows that

$$\begin{aligned}
P^\omega\left(\frac{\overline{X}_t - \mathcal{G}^\omega(t)}{\vartheta\sqrt{t}} < x\right) &= P^\omega\big(\tau_{\mathcal{I}^\omega(t)} > t\big) \\
&= P^\omega\left(\frac{\tau_{\mathcal{I}^\omega(t)} - \mathcal{H}^\omega(\mathcal{I}^\omega(t))}{\sigma\sqrt{\mathcal{I}^\omega(t)}} > \frac{t - \mathcal{H}^\omega(\mathcal{I}^\omega(t))}{\sigma\sqrt{t}} \cdot \sqrt{\frac{t}{\mathcal{I}^\omega(t)}}\right).
\end{aligned}$$

The sequence $\mathcal{I}^\omega(t)$ is increasing in t and diverges; in particular, by the law of large numbers $t/\mathcal{I}^\omega(t)$ converges to ν_β^{-1} for **P**-a.e. ω. The result then follows from Lemma 3.1.5 if $\mathcal{H}^\omega(\mathcal{I}^\omega(t)) = t + \sigma\nu_\beta^{1/2}x\sqrt{t} + o_t$, where $o_t/\sqrt{t}$ converges to 0 for **P**-a.e. ω.

Since $\mathcal{I}^\omega(t)$ diverges, by Lemma 3.1.6 it suffices to show that $\tilde{\mathcal{H}}^\omega(\mathcal{I}^\omega(t)) = t + \sigma\nu_\beta^{1/2}x\sqrt{t} + o_t$. By definition of $\tilde{\mathcal{H}}^\omega$ and $\mathcal{I}^\omega(n)$ we have that there exists some $O_1 := O_1(\omega, t, x)$ such that $|O_1| \leq \nu_\beta^{-1}$ and $\tilde{\mathcal{H}}^\omega(\mathcal{I}^\omega(t))$ is equal to

$$\begin{aligned}
\nu_\beta^{-1}\mathcal{I}^\omega(t) &+ \sum_{k=0}^{\mathcal{I}^\omega(t)-1} \frac{\beta+1}{\beta-1}(E^\omega[\eta_{k,0}] - \mathbb{E}[\eta_{k,0}]) \\
&= t + \sigma\nu_\beta^{1/2}x\sqrt{t} - \sum_{k=0}^{\lfloor\nu_\beta t\rfloor-1} \frac{\beta+1}{\beta-1}(E^\omega[\eta_{k,0}] - \mathbb{E}[\eta_{k,0}]) \\
&\quad + \sum_{k=0}^{\mathcal{I}^\omega(t)-1} \frac{\beta+1}{\beta-1}(E^\omega[\eta_{k,0}] - \mathbb{E}[\eta_{k,0}]) + O_1.
\end{aligned}$$

Moreover, for some $O_2 := O_2(\omega, t, x)$ satisfying $|O_2| \leq 3$, we have that

$$\mathcal{I}^\omega(t) - \lfloor\nu_\beta t\rfloor = \vartheta x\sqrt{t} + \mathbb{E}[\eta_0]^{-1}\sum_{k=0}^{\lfloor\nu_\beta t-1\rfloor}(E^\omega[\eta_{k,0}] - \mathbb{E}[\eta_{k,0}]) + O_2.$$

By part 1 of Lemma 3.1.7 we have that $(\mathcal{I}^\omega(t) - \lfloor\nu_\beta t\rfloor)t^{-\frac{1+c}{2}}$ converges to 0 for **P**-a.e. ω and any $c > 0$. In order to show that $\tilde{\mathcal{H}}^\omega(\mathcal{I}^\omega(t)) = t + \sigma\nu_\beta^{1/2}x\sqrt{t} + o_t$ it now suffices to show that for all $c > 0$ suitably small

$$\mathcal{R}^\omega(n, c) := n^{-1/2}\max_{m\leq n^{\frac{1+c}{2}}}\left|\sum_{k=n}^{n+m}(E^\omega[\eta_{k,0}] - \mathbb{E}[\eta_{k,0}])\right|$$

converges to 0 for $\mathbf{P}$-a.e. ω.

Suppose that $\mathcal{R}^\omega(n^2, 2c)$ converges to 0 for all $c > 0$ suitably small and $\mathbf{P}$-a.e. ω. Then, for $i = 1, ..., 2n$ (that is, $n^2 < n^2 + i < (n+1)^2$) we have that

$$\begin{aligned}
&\mathcal{R}^\omega(n^2+i, c) \\
&= (n^2+i)^{-1/2} \max_{m \le (n^2+i)^{\frac{1+c}{2}}} \left| \sum_{k=n^2+i}^{n^2+i+m} (E^\omega[\eta_{k,0}] - \mathbb{E}[\eta_{k,0}]) \right| \\
&\le (n^2+i)^{-1/2} \max_{m \le (n^2+i)^{\frac{1+c}{2}}} \left(\left| \sum_{k=n^2}^{n^2+i+m} (E^\omega[\eta_{k,0}] - \mathbb{E}[\eta_{k,0}]) \right| + \left| \sum_{k=n^2}^{n^2+i-1} (E^\omega[\eta_{k,0}] - \mathbb{E}[\eta_{k,0}]) \right| \right).
\end{aligned}$$

Since $i + m < n^{1+2c}$ for n suitably large we now have that $\mathcal{R}^\omega(n^2+i, c) \le 2\mathcal{R}^\omega(n^2, 2c)$ for all $i = 1, ..., 2n$ thus it suffices to show that $\mathcal{R}^\omega(n^2, 2c)$ converges to 0 for all $c > 0$ suitably small and $\mathbf{P}$-a.e. ω. For $\epsilon > 0$, by Markov's inequality

$$\begin{aligned}
\mathbf{P}\left(\mathcal{R}^\omega(n^2, 2c) > \epsilon\right) &\le \mathbf{E}\left[\mathcal{R}^\omega(n^2, 2c)^{2+2\delta}\right] \epsilon^{-(2+2\delta)} \\
&= C_\epsilon \mathbf{E}\left[\left(n^{-1} \max_{m \le n^{1+2c}} \left| \sum_{k=n^2}^{n^2+m} (E^\omega[\eta_{k,0}] - \mathbb{E}[\eta_{k,0}]) \right| \right)^{2+2\delta}\right] \\
&\le C_\epsilon n^{(1+2c)(1+\delta) - 2(1+\delta)}
\end{aligned}$$

by part 2 of Lemma 3.1.7. Choosing $c < \delta/(2+2\delta)$ gives us that

$$\sum_{n=1}^\infty \mathbf{P}\left(\mathcal{R}^\omega(n^2, 2c) > \epsilon\right) < \infty$$

therefore by the Borel-Cantelli lemma we have the desired result. □

Remark 3.1.8. *Notice that, since $\nu_\beta t$ is the centring of X_t in the annealed CLT (Theorem 3.2) and $\mathcal{G}^\omega(t) - \nu_\beta t$ is a sum of i.i.d. random variables under the environment law, whenever these random variables have non-zero variance we obtain a central limit theorem for $\mathcal{G}^\omega(t) - \nu_\beta t$ with respect to $\mathbf{P}$ and therefore proving Theorem 3.3 also shows that there is no quenched CLT for X_t with a deterministic centring.*

3.2 Random walks on subcritical Galton-Watson trees

In this section we apply the results of the previous section to prove Theorems 3.4, 3.5 and 3.6. Recall from Section 2.5 that we can couple the random walk on a subcritical GW-tree conditioned to survive with a randomly trapped random walk on $\mathbb{Z}$ so that the walks do not deviate too far from each other. We can, therefore, consider this randomly trapped random walk and it suffices to show that the conditions for Theorems 3.1, 3.2

and 3.3 hold. With this aim, it is enough to prove suitable moment bounds on the excursion times of biased random walks in random trees.

3.2.1 The speed of the walk

We wish to prove a bound on the expected holding time for the randomly trapped random walk. Let η_k be the k^{th} holding time of the randomly trapped random walk. Recall from Section 2.5 that η_0 is distributed as the first hitting time of $\overline{\rho}$ by the walk W_n on the tree $\overline{\mathcal{T}}$ which is a random tree rooted at ρ with a single ancestor $\overline{\rho}$ of the root, $\xi^* - 1$ buds attached as children of ρ (where ξ^* has the size biased law) and independent f GW-trees attached to the buds. Let

$$u_\rho = \sum_{k=1}^{\tau_{\overline{\rho}}} \mathbf{1}_{\{W_k = \rho\}} \tag{3.9}$$

be the number of return times to the root ρ before reaching its unique ancestor $\overline{\rho}$. That is, u_ρ is the number of excursions to the trees attached to ρ before the walk reaches $\overline{\rho}$. Let $\tau_x^{(0)} := 0$ then for $j = 1, ..., u_\rho$ write $\tau_x^{(j)} := \min\{n > \tau_x^{(j-1)} : W_n = x\}$ to be the hitting times of x and $\kappa_j := \tau_\rho^{(j)} - \tau_\rho^{(j-1)}$ the duration of the j^{th} excursion. We then have that

$$\eta_0 \stackrel{\text{d}}{=} 1 + \sum_{j=1}^{u_\rho} \kappa_j. \tag{3.10}$$

Lemma 3.2.1. *Suppose $\beta\mu < 1$, $\sigma^2 < \infty$ and $\beta \geq 1$, then*

$$\mathbb{E}[\eta_0] = \frac{\mu(\beta+1)(1-\beta\mu) + 2\beta(\sigma^2 - \mu(1-\mu))}{\mu(\beta+1)(1-\beta\mu)}.$$

Moreover, if $\sigma^2 = \infty$ or $\beta \geq \mu^{-1}$ then $\mathbb{E}[\eta_0] = \infty$.

Proof. By (3.10) we have that

$$\mathbb{E}[\eta_0] = 1 + \mathbb{E}\left[\sum_{j=1}^{u_\rho} \mathbb{E}[\kappa_j | u_\rho]\right] = 1 + \mathbb{E}[u_\rho]\mathbf{E}\left[E_\rho^{\mathcal{T}^f}[\tau_\rho^+] | Z_1 = 1\right]$$

where Z_1 denotes the first generation size of an f-GW-tree $\mathcal{T}^f$. The number of excursions u_ρ is geometrically distributed under $P_\rho^{\overline{\mathcal{T}}}$ with termination probability $1 - p_{ex}$ where

$$p_{ex} := P_\rho^{\overline{\mathcal{T}}}(W_1 \neq \overline{\rho}) = \frac{\beta(\xi^* - 1)}{\beta\xi^* + 1}.$$

It therefore follows that

$$\mathbb{E}[u_\rho] = \mathbb{E}\left[\frac{\beta(\xi^*-1)}{\beta+1}\right] = \frac{\beta}{\beta+1}\left(\sum_{k=1}^{\infty}\frac{k^2p_k}{\mu}-1\right) = \frac{\beta(\sigma^2-\mu(1-\mu))}{(\beta+1)\mu}.$$

Using the formula (2.7) for the expected time spent in a fixed tree and statement 1 of Lemma 2.4.1 for the expected size of the k^{th} generation we have that

$$\begin{aligned}\mathbf{E}\left[E_\rho^{\mathcal{T}^f}[\tau_\rho^+]|Z_1=1\right] &= \mathbf{E}\left[2\sum_{k\geq 1}\frac{Z_k\beta^{k-1}}{Z_1}|Z_1=1\right] \\ &= 2\sum_{k\geq 1}\mathbf{E}[Z_k|Z_1=1]\beta^{k-1} \\ &= 2\sum_{k\geq 1}(\beta\mu)^{k-1}.\end{aligned}$$

If $\beta<\mu^{-1}$ then this is equal to $2/(1-\beta\mu)$; otherwise, the sum does not converge. It follows that

$$\mathbb{E}[\eta_0] = \frac{\mu(\beta+1)(1-\beta\mu)+2\beta(\sigma^2-\mu(1-\mu))}{\mu(\beta+1)(1-\beta\mu)}.$$

□

Combining this with Theorem 3.1 and the results of Section 2.5 we have that Theorem 3.4 holds. We now extend the Einstein relation for the randomly trapped random walk to the walk on the GW-tree. This is a non-trivial extension because, in the tree model, the bias affects the trapping times and the unbiased walk is significantly influenced by the restriction to the half line. For this reason we observe convergence to a reflected Brownian motion and cannot simply apply Corollary 3.1.2.

Lemma 3.2.2. *Suppose $\mu<1$ and $\sigma^2<\infty$. The unbiased ($\beta=1$) walk $|X_{\lfloor nt\rfloor}|n^{-1/2}$ converges in $\mathbb{P}$-distribution on $D_{J_1}([0,\infty),\mathbb{R})$ as $n\to\infty$ to $|B_t|$ where B_t is a scaled Brownian motion with variance $\Upsilon=\mathbb{E}[\eta_0]^{-1}$. Moreover,*

$$\lim_{\beta\to 1^+}\frac{\nu_\beta}{\beta-1}=\frac{\Upsilon}{2}$$

where ν_β is the speed calculated in Theorem 3.4 for the β-biased walk.

Proof. Recall that $\hat{X}_n$ is a randomly trapped random walk on $\mathbb{Z}$ which, by assumption and Lemma 3.2.1, is unbiased and has finite expected holding times. By [6, Theorem 2.9], for $\mathbf{P}$-a.e. ω, the rescaled process $\hat{X}_{nt}n^{-1/2}$ converges in P^ω-distribution to a scaled Brownian motion B with variance $\mathbb{E}[\eta_0]^{-1}$.

The scaled local time at the origin $n^{-1}\mathcal{L}^{\hat{Y}}(0,n-1)$ converges P-a.s. to 0. Moreover, the holding times $(\eta_{0,i})_{i\geq 1}$ are i.i.d. under P^ω therefore, by the law of large numbers, $\sum_{i=1}^{n} n^{-1}\eta_{0,i}$ converges P^ω-a.s. to $E^\omega[\eta_{0,1}]$ for $\mathbf{P}$-a.e. ω. The same holds for the times between movements along the backbone $(\tilde{\eta}_{0,i})_{i\geq 1}$ therefore the scaled sums

$$\sum_{i=1}^{\mathcal{L}^{\hat{Y}}(0,n-1)} \frac{\eta_{0,i}}{n} \qquad \text{and} \qquad \sum_{i=1}^{\mathcal{L}^{\hat{Y}}(0,n-1)} \frac{\tilde{\eta}_{0,i}}{n}$$

converge to 0, $\mathbb{P}$-a.s. It follows that the process $\overline{X}_n := \hat{Y}_{\overline{S}_n^{-1}}$ where

$$\overline{S}_n := \sum_{x\in\mathbb{Z}} \sum_{i=1}^{\mathcal{L}^{\hat{Y}}(x,n-1)} \overline{\eta}_{x,i} \qquad \text{and} \qquad \overline{\eta}_{x,i} = \begin{cases} \eta_{x,i}, & \text{if } x<0, \\ \tilde{\eta}_{\rho_x,i}, & \text{if } x\geq 0, \end{cases}$$

obeys the same central limit theorem as $\hat{X}$. That is, we may replace the trap at the origin with the slightly different trap which corresponds to the branch at the root of the GW-tree, and still obtain a central limit theorem.

Define the time spent above 0 by $\overline{X}$ and the associated limiting Brownian motion B as

$$A_t^{\overline{X}} := \int_0^t \mathbf{1}_{\{\overline{X}_s\geq 0\}}\mathrm{d}s \qquad \text{and} \qquad A_t^B := \int_0^t \mathbf{1}_{\{B_s\geq 0\}}\mathrm{d}s.$$

By substitution we then have that

$$\frac{A_{tn}^{\overline{X}}}{n} = \int_0^t \mathbf{1}_{\{\overline{X}_{rn}/n^{1/2}\geq 0\}}\mathrm{d}r \;\to\; A_t^B \qquad \text{since} \qquad \lim_{\varepsilon\to 0^+}\int_0^t \mathbf{1}_{\{B_s\in[-\varepsilon,\varepsilon]\}}\mathrm{d}s = 0$$

In particular, $(\overline{X}_{nt}n^{-1/2}, A_t^{\overline{X}}n^{-1})_{t\geq 0}$ converges to $(B_t, A_t^B)_{t\geq 0}$. Recall that $\tilde{X}$ is the projection of X onto the backbone. By definition of $A_t^{\overline{X}}$, we have that $|\tilde{X}_{nt}|n^{-1/2} = \overline{X}_{(A_{tn}^{\overline{X}})^{-1}}n^{-1/2}$ which converges in distribution to $B_{(A_t^B)^{-1}}$ by Proposition 2.1.1. Furthermore, $B_{(A_t^B)^{-1}}$ is equal in distribution to $|B_t|$ hence we have the desired convergence result by Section 2.5.

Using Theorem 3.4 and taking the limit as $\beta\to 1^+$ we have that

$$\frac{\nu_\beta}{\beta-1} \to \frac{\mu(1-\mu)}{2\sigma^2} = \frac{1}{2\mathbb{E}[\eta_0]}$$

by Lemma 3.2.1, which completes the proof. □

3.2.2 An annealed functional central limit theorem

We now prove an annealed functional central limit theorem for the biased walk on the subcritical GW-tree conditioned to survive. By Lemmas 2.5.1 and 2.5.2 it will suffice to show the result holds for the corresponding randomly trapped random walk. We obtain the result by using the annealed invariance principle Theorem 3.2. That is, we show conditions on the tree and the bias which ensure that $\mathbb{E}[\eta_0^2] < \infty$.

Recall from (3.9) that u_ρ is the number of return times to the root ρ before reaching its unique ancestor $\overline{\rho}$ and κ_j is the duration of the j^{th} such excursion. Letting $\overline{\mathcal{T}} = \overline{\mathcal{T}}_0$ denote the first branch, by (3.10) we have that

$$E^\omega\left[\eta_0^2\right] = E^{\overline{\mathcal{T}}}\left[\sum_{j=1}^{u_\rho}\kappa_j^2 + \sum_{j=1}^{u_\rho}\sum_{i\neq j}\kappa_j\kappa_i + 2\sum_{j=1}^{u_\rho}\kappa_j + 1\right].$$

We want to show that $\mathbf{E}\left[E^\omega\left[\eta_0^2\right]\right] < \infty$. Lemma 3.2.3 shows that this can be reduced to showing that the expected value of the first sum in the quenched expectation is finite.

Lemma 3.2.3. *If $\beta > 1$, $\beta^2\mu < 1$ and $\mathbf{E}[\xi^3] < \infty$ then*

$$\mathbf{E}\left[E^{\overline{\mathcal{T}}}\left[\sum_{j=1}^{u_\rho}\sum_{i\neq j}\kappa_j\kappa_i + 2\sum_{j=1}^{u_\rho}\kappa_j + 1\right]\right] < \infty.$$

Proof. By (3.10) we have that

$$E^{\overline{\mathcal{T}}}\left[2\sum_{j=1}^{u_\rho}\kappa_j + 1\right] < 2E^\omega[\eta_0]$$

which has finite expectation under $\mathbf{P}$ by Lemma 3.2.1 since $\beta^2\mu < 1$ implies that $\beta\mu < 1$.

The variable u_ρ is geometrically distributed with termination probability $1-p_{ex}$; that is,

$$P^{\overline{\mathcal{T}}}(u_\rho = n) = p_{ex}^n(1-p_{ex}) \quad \text{where} \quad p_{ex} = \frac{\beta(\xi^*-1)}{\beta\xi^*+1} \tag{3.11}$$

and ξ^*+1 is the number of neighbours of ρ in $\overline{\mathcal{T}}$. We then have that

$$\begin{aligned} E^{\overline{\mathcal{T}}}\left[\sum_{j=1}^{u_\rho}\sum_{i\neq j}\kappa_j\kappa_i\right] &= \sum_{n=0}^{\infty} P^{\overline{\mathcal{T}}}(u_\rho = n)\sum_{j=1}^{n}\sum_{i\neq j}E^{\overline{\mathcal{T}}}[\kappa_j\kappa_i] \\ &= E^{\overline{\mathcal{T}}}[\kappa_1]^2\sum_{n=0}^{\infty} n(n-1)p_{ex}^n(1-p_{ex}) \\ &= 2\left(\frac{\beta}{\beta+1}\right)^2 E^{\overline{\mathcal{T}}}[\kappa_1]^2(\xi^*-1)^2 \end{aligned} \tag{3.12}$$

by independence of excursion times under $P^{\overline{\mathcal{T}}}$.

Let $\overline{\mathcal{T}}^\circ$ denote the tree $\overline{\mathcal{T}}$ without the ancestor of the root $\overline{\rho}$ and $\mathcal{T}^f$ be an

f-GW-tree. Write Z_n and Z_n^f to be the n^{th} generation sizes of $\overline{\mathcal{T}}^{\circ}$ and $\mathcal{T}^f$ respectively then for $j = 1, 2, ...$ let $Z_n^{f,j}$ be independent copies of Z_n^f. By the construction of $\overline{\mathcal{T}}^{\circ}$ using GW-trees we have that

$$\begin{aligned}
\sum_{l=1}^{\infty} l^2 \mathbf{P}\left(Z_k = l | Z_1 = i\right) &= \mathbf{E}\left[(Z_{k-1}^f)^2 | Z_0^f = i\right] \\
&= \mathbf{E}\left[\left(\sum_{j=1}^{i} Z_{k-1}^{f,j}\right)^2\right] \\
&= i\mathbf{E}\left[(Z_{k-1}^f)^2\right] + i(i-1)\mathbf{E}\left[Z_{k-1}^f\right]^2.
\end{aligned}$$

Using this with Lemma 2.4.1, for $j \geq k \geq 1$ we have that

$$\begin{aligned}
\mathbf{E}\left[Z_k Z_j\right] &= \sum_{i=1}^{\infty} \mathbf{P}\left(\xi^* - 1 = i\right) \sum_{l=1}^{\infty} l \mathbf{P}\left(Z_k = l | Z_1 = i\right) \mathbf{E}\left[Z_j | Z_k = l\right] \\
&= \sum_{i=1}^{\infty} \mathbf{P}\left(\xi^* - 1 = i\right) \sum_{l=1}^{\infty} l^2 \mathbf{P}\left(Z_k = l | Z_1 = i\right) \mathbf{E}\left[Z_{j-k}^f\right] \\
&= \mu^{j-k} \sum_{i=1}^{\infty} \mathbf{P}\left(\xi^* - 1 = i\right) \left(i\mathbf{E}\left[(Z_{k-1}^f)^2\right] + i(i-1)\mathbf{E}\left[Z_{k-1}^f\right]^2\right) \\
&\leq C\mu^j \mathbf{E}[(\xi^* - 1)^2]. \qquad (3.13)
\end{aligned}$$

Using (3.12) and the formula (2.7) for the expected time spent in a tree we have that

$$\begin{aligned}
\mathbf{E}\left[E^{\overline{\mathcal{T}}}\left[\sum_{j=1}^{u_\rho} \sum_{i \neq j} \kappa_j \kappa_i\right]\right] &= C_\beta \mathbf{E}\left[(\xi^* - 1)^2 E^{\overline{\mathcal{T}}}[\kappa_1]^2\right] \\
&= C_\beta \mathbf{E}\left[\left(\sum_{k \geq 1} \beta^k Z_k\right)^2\right] \\
&= C_\beta \sum_{k \geq 1} \beta^{2k} \mathbf{E}[Z_k^2] + 2C_\beta \sum_{k \geq 1} \sum_{j > k} \beta^{k+j} \mathbf{E}[Z_k Z_j].
\end{aligned}$$

By (3.13) we then have that

$$\sum_{k \geq 1} \sum_{j > k} \beta^{k+j} \mathbf{E}[Z_k Z_j] \;\leq\; C\mathbf{E}[(\xi^* - 1)^2] \sum_{k \geq 1} \beta^k \sum_{j > k} (\beta\mu)^j \;\leq\; C\mathbf{E}[(\xi^* - 1)^2] \sum_{k \geq 1} (\beta^2\mu)^k$$

and

$$\sum_{k \geq 1} \beta^{2k} \mathbf{E}[Z_k^2] \;\leq\; C\mathbf{E}[(\xi^* - 1)^2] \sum_{k \geq 1} (\beta^2\mu)^k.$$

Each of these terms is finite since $\beta^2\mu < 1$ and $\mathbf{E}[\xi^3] < \infty$ where we recall from the definition of the size-biased distribution that $\mathbf{E}[(\xi^*-1)^2] \le C\mathbf{E}[\xi^3]$. $\square$

Let $\mathscr{T}$ be a fixed tree and $(X_n)_{n\ge1}$ a β-biased walk on $\mathscr{T}$. For $x \in \mathscr{T}$ recall that

$$v_x := \sum_{k=1}^{\tau_\rho^+} \mathbf{1}_{\{X_k=x\}}$$

denotes the number of visits to x before returning to the root; then $\tau_\rho^+ = \sum_{x\in\mathscr{T}} v_x$ and

$$E_\rho^{\mathscr{T}}\left[(\tau_\rho^+)^2\right] = \sum_{x,y\in\mathscr{T}} E_\rho^{\mathscr{T}}[v_x v_y]. \tag{3.14}$$

For any $x, y \in \mathscr{T}$ there exists a unique vertex $w_{x,y}$ which is the closest ancestor of both x and y. Moreover, by Lemma 2.3.6,

$$E_\rho^{\mathscr{T}}[v_x v_y] \le E_{w_{x,y}}^{\mathscr{T}}[v_x v_y] \le C_\beta(|c(x)|\beta+1)(|c(y)|\beta+1)\beta^{|x|+|y|}$$

where $c(x)$ is the set of children of x in $\mathscr{T}$.

In order to show that $\mathbb{E}[\eta_0^2] < \infty$ it remains to prove Lemma 3.2.4 which follows similarly to Lemma 3.2.3 with the use of Lemma 2.3.6.

Lemma 3.2.4. *If $\beta > 1$, $\beta^2\mu < 1$ and $\mathbf{E}[\xi^3] < \infty$ then*

$$\mathbf{E}\left[E^{\overline{\mathcal{T}}}\left[\sum_{j=1}^{u_\rho} \kappa_j^2\right]\right] < \infty.$$

Proof. Recall that $\overline{\mathcal{T}}^\circ$ denotes the tree $\overline{\mathcal{T}}$ without the ancestor of the root $\overline{\rho}$. Since the separate excursions are independent under $P^{\overline{\mathcal{T}}}$ and u_ρ is geometrically distributed we have that

$$\mathbf{E}\left[E^{\overline{\mathcal{T}}}\left[\sum_{j=1}^{u_\rho} \kappa_j^2\right]\right] = \mathbf{E}\left[\left(\frac{\beta(\xi^*-1)}{\beta+1}\right) E_\rho^{\overline{\mathcal{T}}^\circ}\left[\kappa_1^2\right]\right].$$

Labelling $\rho_1, ..., \rho_{\xi^*-1}$ as the neighbours of ρ in $\overline{\mathcal{T}}^\circ$, and $\overline{\mathcal{T}}_{\rho_j}$ as the tree consisting of ρ, ρ_j and the descendants of ρ_j we have that

$$E_\rho^{\overline{\mathcal{T}}^\circ}\left[\kappa_1^2\right] = \sum_{j=1}^{\xi^*-1} \frac{E_\rho^{\overline{\mathcal{T}}_{\rho_j}}\left[(\tau_\rho^+)^2\right]}{\xi^*-1}$$

when $\xi^* \ne 1$ and 0 otherwise. In particular, it then follows that

$$\mathbf{E}\left[E^{\overline{\mathcal{T}}}\left[\sum_{j=1}^{u_\rho} \kappa_j^2\right]\right] \le \mathbf{E}\left[\sum_{j=1}^{\xi^*-1} E^{\overline{\mathcal{T}}_{\rho_j}}\left[(\tau_\rho^+)^2\right]\right] = \mathbf{E}[\xi^*-1]\mathbf{E}\left[E^{\overline{\mathcal{T}}_{\rho_1}}\left[(\tau_\rho^+)^2\right]\right]$$

since the subtraps are independent. Since $\mathbf{E}[\xi^*-1] \leq C\mathbf{E}[\xi^2] < \infty$, it suffices to show that

$$\mathbf{E}\left[E_\rho^{\widetilde{\mathcal{T}}}\left[(\tau_\rho^+)^2\right]\right] < \infty$$

where $\widetilde{\mathcal{T}}$ is a tree (equal in distribution to $\overline{\mathcal{T}}_{\rho_1}$) with root ρ, single first generation vertex $\overrightarrow{\rho}$ and, under $\mathbf{P}$, the subtree rooted at $\overrightarrow{\rho}$ is an f-GW-tree.

Recall that $\widetilde{\mathcal{T}}_z$ denotes the descendent tree of $\widetilde{\mathcal{T}}$ at z. By (3.14) and Lemma 2.3.6 we have that

$$\begin{aligned}
\mathbf{E}\left[E_\rho^{\widetilde{\mathcal{T}}}\left[(\tau_\rho^+)^2\right]\right] &= \mathbf{E}\left[\sum_{x,y\in\widetilde{\mathcal{T}}} E_\rho^{\widetilde{\mathcal{T}}}[v_x v_y]\right] \\
&\leq C_\beta \mathbf{E}\left[\left(\sum_{x\in\widetilde{\mathcal{T}}}\left(|c(x)|\beta+1\right)\beta^{|x|}\right)\left(\sum_{y\in\widetilde{\mathcal{T}}}\left(|c(y)|\beta+1\right)\beta^{|y|}\right)\right].
\end{aligned}$$

By collecting terms in the k^{th} generation we have that

$$\sum_{x\in\widetilde{\mathcal{T}}}\left(|c(x)|\beta+1\right)\beta^{|x|} \;=\; 1+\sum_{k\geq 1} Z_k^{\widetilde{\mathcal{T}}}(\beta^k+\beta^{k-1}) \;\leq\; (1+\beta^{-1})\sum_{k\geq 0} Z_k^{\widetilde{\mathcal{T}}}\beta^k$$

where $Z_k^{\widetilde{\mathcal{T}}}$ is the size of the k^{th} generation of $\widetilde{\mathcal{T}}$. For $k \geq 0$ the tree $\widetilde{\mathcal{T}}$ satisfies $Z_{k+1}^{\widetilde{\mathcal{T}}} = Z_k$ for a GW-process Z_k with $Z_0 = 1$; therefore, using that $\beta^2\mu < 1$ and Lemma 2.4.1, we have that $\mathbf{E}\left[Z_k^{\widetilde{\mathcal{T}}} Z_j^{\widetilde{\mathcal{T}}}\right] \leq C\mu^j$, for $j \geq k$. In particular,

$$\mathbf{E}\Big[E_\rho^{\widetilde{\mathcal{T}}}\left[(\tau_\rho^+)^2\right]\Big] \;\leq\; C_\beta \sum_{k\geq 0}\beta^k\sum_{j\geq k}\beta^j\mathbf{E}\Big[Z_k^{\widetilde{\mathcal{T}}} Z_j^{\widetilde{\mathcal{T}}}\Big] \;\leq\; C_\beta\sum_{k\geq 0}\beta^k\sum_{j\geq k}(\mu\beta)^j \;\leq\; C_{\beta,\mu}\sum_{k\geq 0}(\mu\beta^2)^k$$

which is finite since $\mu\beta^2 < 1$. ☐

By Lemmas 3.2.3 and 3.2.4 we have that $\mathbb{E}[\eta_0^2] < \infty$ therefore, by Lemmas 2.5.1 and 2.5.2 and Theorem 3.2, we have that Theorem 3.5 holds.

Remark 3.2.5. *Recall that the expression (3.8) for ς^2 was given in Theorem 3.2 in terms of the moments of the distance and time between regenerations. We can therefore use this to write the corresponding form in the GW-tree model as*

$$\varsigma^2 = \frac{\mathbb{E}\left[\left(\varrho_2-\varrho_1-\nu_\beta\sum_{j=\zeta_1^Y}^{\zeta_2^Y-1}\eta_j\right)^2\right]}{\mathbb{E}[\eta_0]\mathbb{E}[\zeta_2^Y-\zeta_1^Y]}$$

where ζ_j^Y and ϱ_j are the regeneration times and points of the walk Y.

We now show that both of the conditions $\beta^2\mu < 1$ and $\mathbf{E}[\xi^3] < \infty$ are necessary in order to apply Theorem 3.2. This suggests that we should only have an annealed functional central limit theorem for the walk on the subcritical GW-tree conditioned to survive when these conditions hold.

Lemma 3.2.6. *If $\beta^2\mu \geq 1$ or $\mathbf{E}[\xi^3] = \infty$ then*

$$\mathbb{E}[\eta_1^2] \geq \mathbf{E}\left[E^{\overline{\mathcal{T}}}[\eta_1]^2\right] = \infty.$$

Proof. Recall that η_1 is the first hitting time of $\overline{\rho}$ by W_n started from the root ρ in $\overline{\mathcal{T}}$. With positive probability ρ has neighbours other than $\overline{\rho}$ and the walk moves to one on the first step. Until returning to ρ the walk is equal in distribution to a β-biased random walk on an f-GW-tree conditioned to have a single first generation vertex. In particular, it suffices to show that for a β-biased walk

$$\mathbf{E}\left[E_\rho^{\mathcal{T}^f}\left[\tau_\rho^+\right]^2 |Z_1 = 1\right] = \infty$$

where $\mathcal{T}^f$ is an f GW-tree rooted at ρ. Using the formula for the expected time spent in a tree (2.7) we have that

$$\mathbf{E}\left[E_\rho^{\mathcal{T}^f}\left[\tau_\rho^+\right]^2 |Z_1 = 1\right] = \frac{4}{\beta^2}\mathbf{E}\left[\left(\sum_{k\geq 1}\beta^k Z_k\right)^2 |Z_1 = 1\right] \geq \frac{4}{\beta^2}\sum_{k\geq 1}\beta^{2k}\mathbf{E}[Z_k^2|Z_1 = 1].$$

Since Z_k takes nonnegative values in $\mathbb{Z}$ we have that

$$\mathbf{E}\left[Z_k^2|Z_1 = 1\right] \geq \mathbf{E}\left[Z_k|Z_1 = 1\right] = \mu^{k-1}$$

by statement 1 of Lemma 2.4.1. We therefore have that

$$\mathbf{E}\left[E_\rho^{\mathcal{T}^f}[\tau_\rho^+]^2|Z_1 = 1\right] \geq c\sum_{k\geq 1}(\beta^2\mu)^k$$

which is infinite if $\beta^2\mu \geq 1$.

The first hitting time of $\overline{\rho}$ is at least the number of visits to the offspring of ρ. From ρ, the walk takes a geometric number of visits (with termination probability $1 - p_{ex}$, see (3.11)) to these vertices before reaching $\overline{\rho}$. Using properties of geometric random variables we then have that

$$\mathbf{E}\left[E_\rho^{\overline{\mathcal{T}}}[\eta_1]^2\right] \;\geq\; \mathbf{E}\left[\left(\frac{(\xi^* - 1)\beta}{\beta + 1}\right)^2\right] \;\geq\; c\left(\mathbf{E}[(\xi^*)^2] - 1\right)$$

which is infinite if $\mathbf{E}[\xi^3] = \infty$. □

3.2.3 A quenched central limit theorem

We now prove a quenched central limit theorem for the biased walk on the subcritical GW-tree conditioned to survive. As in the annealed case, by Lemmas 2.5.1 and 2.5.2 it will suffice to show the result holds for the corresponding randomly trapped random walk and we obtain the result by using Theorem 3.3.

Define

$$\mathcal{G}^{\mathcal{T}}(t) = \nu_\beta t - \nu_\beta \sum_{k=1}^{\lfloor \nu_\beta t \rfloor} \frac{\beta+1}{\beta-1}\left(E^{\mathcal{T}}[\tilde{\eta}_{\rho_k,0}] - \mathbb{E}[\eta_0]\right).$$

We want to show that if $\beta^2\mu < 1$ and $\mathbf{E}[\xi^{3+\delta}] < \infty$ for some $\delta > 0$ then

$$\mathbf{E}\left[E^{\overline{\mathcal{T}}}\left[\eta_1\right]^{2+\varepsilon}\right] < \infty$$

for some $\varepsilon > 0$. By Theorems 3.3 and 3.5 it then follows that there exists $\vartheta > 0$ such that for $\mathbf{P}$-a.e. $\mathcal{T}$ we have that

$$P^{\mathcal{T}}\left(\frac{|X_t| - \mathcal{G}^{\mathcal{T}}(t)}{\vartheta\sqrt{t}} \leq x\right) \to \Phi(x) \ := \ \int_{-\infty}^{x} \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} \mathrm{d}u$$

uniformly in x as $n \to \infty$.

Proof of Theorem 3.6. Recall that

$$u_\rho = \sum_{n=1}^{\tau^+_{\overleftarrow{\rho}}} \mathbf{1}_{\{W_n=\rho\}}$$

is the number of hitting times of the root ρ before reaching $\overleftarrow{\rho}$ (for the walk started at ρ) and $\overline{\mathcal{T}}^{\circ}$ is the tree $\overline{\mathcal{T}}$ with $\overleftarrow{\rho}$ removed. Then,

$$E^{\overline{\mathcal{T}}}_\rho[\eta_1] \ = \ 1+E^{\overline{\mathcal{T}}}[u_\rho]E^{\overline{\mathcal{T}}^{\circ}}_\rho\left[\tau^+_\rho\right] \ = \ 1+2E^{\overline{\mathcal{T}}}[u_\rho]\sum_{n\geq 1}\frac{Z_n}{Z_1}\beta^{n-1} \ \leq \ 2(E^{\overline{\mathcal{T}}}[u_\rho]+1)\sum_{n\geq 1}\frac{Z_n}{Z_1}\beta^{n-1}$$

by (2.7) where Z_n is the n^{th} generation size of $\overline{\mathcal{T}}^{\circ}$ since the walk on $\overline{\mathcal{T}}^{\circ}$ is β-biased.

For a fixed tree, u_ρ is geometrically distributed with excursion probability p_{ex} (see (3.11)) therefore $E^{\overline{\mathcal{T}}}[u_\rho] \leq Z_1$. By conditioning on Z_1 we therefore have that

$$\begin{aligned}\mathbf{E}\left[E^{\overline{\mathcal{T}}}\left[\eta_1\right]^{2+\varepsilon}\right] &\leq C\mathbf{E}\left[Z_1^{2+\varepsilon}\mathbf{E}\left[\left(\sum_{n\geq 1}\frac{Z_n}{Z_1}\beta^{n-1}\right)^{2+\varepsilon}\Bigg|Z_1\right]\right] \\ &= C\mathbf{E}\left[\left(\sum_{n\geq 1}Z_n\beta^{n-1}\right)^{2+\varepsilon}\right].\end{aligned}$$

We can write

$$Z_n = \sum_{j=1}^{Z_1} Z_{n-1}^{(j)}$$

where $Z^{(j)}$ are independent GW-processes therefore, by convexity,

$$\begin{aligned}
\mathbf{E}\left[\left(\sum_{n\geq 1} Z_n\beta^{n-1}\right)^{2+\varepsilon}\right] &= \mathbf{E}\left[Z_1^{2+\varepsilon}\left(\sum_{j=1}^{Z_1}\sum_{n\geq 1}\frac{Z_{n-1}^{(j)}}{Z_1}\beta^{n-1}\right)^{2+\varepsilon}\right] \\
&\leq \mathbf{E}\left[Z_1^{1+\varepsilon}\sum_{j=1}^{Z_1}\left(\sum_{n\geq 1} Z_{n-1}^{(j)}\beta^{n-1}\right)^{2+\varepsilon}\right] \\
&= \mathbf{E}[(\xi^*-1)^{2+\varepsilon}]\mathbf{E}\left[\left(\sum_{n\geq 1} Z_{n-1}^{(1)}\beta^{n-1}\right)^{2+\varepsilon}\right].
\end{aligned}$$

By the assumptions of the theorem we have that $\mathbf{E}[(\xi^*-1)^{2+\varepsilon}] \leq \mu^{-1}\mathbf{E}[\xi^{3+\varepsilon}] < \infty$ whenever $\varepsilon < \delta$ thus it suffices to show that

$$\mathbf{E}\left[\left(\sum_{n\geq 0} Z_n\beta^n\right)^{2+\varepsilon}\right] < \infty$$

where Z_n now denotes the n^{th} generation size of an f-GW-process.

For $\varepsilon < \delta$, by conditioning on the height $\mathcal{H} := \max\{n \geq 0 : Z_n > 0\}$ of the tree we have that

$$\begin{aligned}
\mathbf{E}\left[\left(\sum_{n\geq 0} Z_n\beta^n\right)^{2+\varepsilon}\right] &\leq \mathbf{E}\left[\beta^{(2+\varepsilon)\mathcal{H}}\mathbf{E}\left[\left(\sum_{n\geq 0} Z_n\right)^{2+\varepsilon}\Big|\mathcal{H}\right]\right] \\
&\leq \mathbf{E}\left[\beta^{(2+\varepsilon)\mathcal{H}}(\mathcal{H}+1)^{2+\varepsilon}\mathbf{E}\left[\max_{n\leq\mathcal{H}} Z_n^{2+\varepsilon}\Big|\mathcal{H}\right]\right] \\
&\leq \mathbf{E}\left[\beta^{(2+\varepsilon)\mathcal{H}}(\mathcal{H}+1)^{2+\varepsilon}\sum_{n=0}^{\mathcal{H}}\mathbf{E}\left[Z_n^{2+\varepsilon}\Big|\mathcal{H}\right]\right] \\
&= \sum_{n=0}^{\infty}\mathbf{E}\left[\beta^{(2+\varepsilon)\mathcal{H}}(\mathcal{H}+1)^{2+\varepsilon}Z_n^{2+\varepsilon}\right] \\
&= \sum_{n=0}^{\infty}\sum_{j=1}^{\infty} j^{2+\varepsilon}\mathbf{P}(Z_n=j)\mathbf{E}\left[\beta^{(2+\varepsilon)\mathcal{H}}(\mathcal{H}+1)^{2+\varepsilon}|Z_n=j\right].
\end{aligned} \tag{3.15}$$

From (2.9) we have that $\mathbf{P}(\mathcal{H} \geq n) \sim c\mu^n$ therefore $\mathbf{P}(\mathcal{H} \geq n) \leq C\mu^n$ for some constant C hence

$$\mathbf{E}\left[\beta^{(2+\varepsilon)\mathcal{H}}(\mathcal{H}+1)^{2+\varepsilon}|Z_n=j\right] = \mathbf{E}\left[\beta^{(2+\varepsilon)(\mathcal{H}+n+1)}(\mathcal{H}+n)^{2+\varepsilon}|Z_0=j\right]$$

$$
\begin{aligned}
&= \sum_{i=1}^{\infty} \beta^{(2+\varepsilon)(i+n)} (i+n+1)^{2+\varepsilon} \mathbf{P}(\mathcal{H} = i | Z_0 = j) \\
&\leq \sum_{i=1}^{\infty} \beta^{(2+\varepsilon)(i+n)} (i+n+1)^{2+\varepsilon} \mathbf{P}(\mathcal{H} \geq i | Z_0 = j) \\
&\leq C \sum_{i=1}^{\infty} \beta^{(2+\varepsilon)(i+n)} (i+n+1)^{2+\varepsilon} j \mu^i \\
&\leq C j \beta^{(2+\varepsilon)n} (n+2)^{2+\varepsilon} \sum_{i=1}^{\infty} i^{2+\varepsilon} (\beta^{2+\varepsilon}\mu)^i.
\end{aligned}
$$

Since $\beta^2\mu < 1$ we can choose $\varepsilon > 0$ sufficiently small so that $\beta^{2+\varepsilon}\mu < 1$ therefore

$$
\sum_{i=1}^{\infty} i^{2+\varepsilon} (\beta^{2+\varepsilon}\mu)^i < \infty.
$$

Substituting back into (3.15) we have that

$$
\begin{aligned}
\mathbf{E}\left[\left(\sum_{n\geq 0} Z_n \beta^n\right)^{2+\varepsilon}\right] &\leq C \sum_{n=0}^{\infty} \beta^{n(2+\varepsilon)} (n+2)^{2+\varepsilon} \sum_{j=1}^{\infty} j^{3+\varepsilon} \mathbf{P}(Z_n = j) \\
&= C \sum_{n=0}^{\infty} \beta^{n(2+\varepsilon)} (n+2)^{2+\varepsilon} \mathbf{E}[Z_n^{3+\varepsilon}]. \qquad (3.16)
\end{aligned}
$$

Using a telescoping sum we can write

$$
Z_n = \mu^n + \sum_{k=0}^{n-1} (Z_{n-k} - \mu Z_{n-(k+1)}) \mu^k,
$$

therefore, by convexity, we have that

$$
\begin{aligned}
\mathbf{E}[Z_n^{3+\varepsilon}] &= (n+1)^{3+\varepsilon} \mathbf{E}\left[\left(\frac{\mu^n}{n+1} + \sum_{k=0}^{n-1} \frac{(Z_{n-k} - \mu Z_{n-(k+1)})\mu^k}{n+1}\right)^{3+\varepsilon}\right] \qquad (3.17) \\
&\leq (n+1)^{3+\varepsilon} \mathbf{E}\left[\frac{\mu^{n(3+\varepsilon)}}{n+1} + \sum_{k=0}^{n-1} \frac{\left((Z_{n-k} - \mu Z_{n-(k+1)})\mu^k\right)^{3+\varepsilon}}{n+1}\right] \\
&= (n+1)^{2+\varepsilon} \mu^{n(3+\varepsilon)} + (n+1)^{2+\varepsilon} \sum_{k=0}^{n-1} \mu^{k(3+\varepsilon)} \mathbf{E}\left[\left(Z_{n-k} - \mu Z_{n-(k+1)}\right)^{3+\varepsilon}\right].
\end{aligned}
$$

Let ξ_j be independent copies of ξ then using the Marcinkiewicz-Zygmund inequality and convexity we have that

$$
\mathbf{E}\Big[\left(Z_{n-k} - \mu Z_{n-(k+1)}\right)^{3+\varepsilon}\Big] = \mathbf{E}\left[\mathbf{E}\left[\left(\sum_{j=1}^{Z_{n-(k+1)}} (\xi_j - \mu)\right)^{3+\varepsilon} \Bigg| Z_{n-(k+1)}\right]\right]
$$

$$
\begin{aligned}
&\leq C\mathbf{E}\left[\mathbf{E}\left[\left(\sum_{j=1}^{Z_{n-(k+1)}}(\xi_j-\mu)^2\right)^{\frac{3+\varepsilon}{2}}\Big|Z_{n-(k+1)}\right]\right]\\
&= C\mathbf{E}\left[\mathbf{E}\left[\left(\sum_{j=1}^{Z_{n-(k+1)}}\frac{(\xi_j-\mu)^2}{Z_{n-(k+1)}}\right)^{\frac{3+\varepsilon}{2}}\Big|Z_{n-(k+1)}\right]Z_{n-(k+1)}^{\frac{3+\varepsilon}{2}}\right]\\
&\leq C\mathbf{E}\left[\mathbf{E}\left[\sum_{j=1}^{Z_{n-(k+1)}}\frac{|\xi_j-\mu|^{3+\varepsilon}}{Z_{n-(k+1)}}\Big|Z_{n-(k+1)}\right]Z_{n-(k+1)}^{\frac{3+\varepsilon}{2}}\right]\\
&= C\mathbf{E}\big[|\xi-\mu|^{3+\varepsilon}\big]\,\mathbf{E}\left[Z_{n-(k+1)}^{\frac{3+\varepsilon}{2}}\right]\\
&\leq C\mathbf{E}\big[|\xi-\mu|^{3+\varepsilon}\big]\,\mathbf{E}\left[Z_{n-(k+1)}^{2}\right].
\end{aligned}
$$

By Lemma 2.4.1 we have that $\mathbf{E}\left[Z_{n-(k+1)}^2\right] \leq C\mu^{n-(k+1)}$ where C is independent of n, k therefore substituting into (3.17) we have that

$$
\mathbf{E}[Z_n^{3+\varepsilon}] \;\leq\; (n+1)^{2+\varepsilon}\mu^{n(3+\varepsilon)} + C(n+1)^{2+\varepsilon}\mu^n\sum_{k=0}^{n-1}\mu^{k(2+\varepsilon)} \;\leq\; C(n+1)^{2+\varepsilon}\mu^n.
$$

Combining with (3.16) we then have that

$$
\mathbf{E}\left[\left(\sum_{n\geq 0}Z_n\beta^n\right)^{2+\varepsilon}\right] \leq C\sum_{n=1}^{\infty}(n+2)^{4+2\varepsilon}(\beta^{2+\varepsilon}\mu)^n
$$

which is finite since we have chosen $\varepsilon > 0$ sufficiently small so that $\beta^{2+\varepsilon}\mu < 1$. $\square$

Remark 3.2.7. *Notice that, under the assumptions of Theorem 3.6, $\nu_\beta t - \mathcal{G}^{\mathcal{T}}(t)$ is a sum of i.i.d. centred random variables with positive, finite variance. It therefore follows that this expression, scaled by $\sqrt{t}$, converges in distribution with respect to* **P** *to a Gaussian random variable. In particular, this means that the environment dependent centring is necessary in the quenched CLT.*

Chapter 4

Escape rates in sub-ballistic phases

In this chapter we introduce the three transient, sub-ballistic regimes for the biased random walk on the subcritical GW-tree conditioned to survive. We then prove the four main theorems concerning these regimes.

Recall from Section 1.2.2 that the subcritical GW-tree conditioned to survive consists of a semi-infinite path $\mathcal{Y}$ (the backbone) with finite trees (called branches) attached to each vertex of $\mathcal{Y}$. In particular, the first generation of the branch (called the buds) has a size-biased distribution $\xi^* - 1$ where $\mathbf{P}(\xi^* = k) = kp_k\mu^{-1}$. It follows from this that for any real valued function φ we have that

$$\mathbf{E}[\varphi(\xi^*)] = \sum_{k=1}^{\infty} \varphi(k)\frac{kp_k}{\mu} = \mathbf{E}[\varphi(\xi)\xi]\mu^{-1}$$

which will prove to be an important property relating the size biased and offspring distributions. Choosing φ to be the identity, we have finite mean of the size-biased distribution if and only if the variance of the offspring distribution is finite. This causes a phase transition for the walk that is not seen in the supercritical tree. The reason for this is that in the corresponding decomposition for the supercritical tree we have subcritical GW-trees as leaves but the number of buds is exponentially tilted and therefore maintains moment properties.

If the offspring law belongs to the domain of attraction of some stable law of index $\alpha \in (1, 2)$ then taking $\varphi(x) = x\mathbf{1}_{\{x \leq t\}}$ shows that the size biased distribution belongs to the domain of attraction of some stable law with index $\alpha - 1$ and allows us to attain properties of the scaling sequences (see Section 2.2).

The first case we consider is when $\beta\mu < 1$ but $\sigma^2 = \infty$; we refer to this as the infinite variance, finite excursion case:

Definition 4.0.1. *(IVFE) The offspring distribution has mean μ satisfying $1 < \beta <$*

μ^{-1} *and belongs to the domain of attraction of a stable law of index* $\alpha \in (1,2)$.

Under this assumption we let L vary slowly at ∞ such that as $x \to \infty$

$$\mathbb{E}\left[\xi^2 \mathbf{1}_{\{\xi \leq x\}}\right] \sim x^{2-\alpha} L(x) \tag{4.1}$$

and choose $(a_n)_{n\geq 1}$ to be a scaling sequence for the size-biased law such that, for any $x > 0$, as $n \to \infty$ we have $\mathbf{P}(\xi^* \geq x a_n) \sim x^{-(\alpha-1)} n^{-1}$. Moreover for some slowly varying function $\tilde{L}$ we have that $a_n = n^{\frac{1}{\alpha-1}} \tilde{L}(n)$.

In this case we have that the slowing is caused by the number of excursions into the branch. Since β is small (i.e. less than μ^{-1}) we have that the expected time spent in a trap (a subcritical GW-tree rooted at a bud) is finite. The number of excursions the walk takes into a branch is of the same order as the number of buds; since the size-biased law has infinite mean there are a large number of buds and, therefore, a large number of excursions. The main result for IVFE is Theorem 4.1 which reflects that Δ_n scales similarly to the sum of independent copies of ξ^*.

Theorem 4.1. *For IVFE, the laws of the process*

$$\left(\frac{\Delta_{nt}}{a_n}\right)_{t\geq 0}$$

converge weakly as $n \to \infty$ *under* $\mathbb{P}$ *on* $D_{M_1}([0,\infty), \mathbb{R})$ *to the law of an* $\alpha - 1$ *stable subordinator* R_t *with Laplace transform*

$$\varphi_t(s) := \mathbb{E}[e^{-sR_t}] = e^{-C_{\alpha,\beta,\mu} t s^{\alpha-1}}$$

where $C_{\alpha,\beta,\mu}$ *is a constant which we shall determine during the proof (see (4.67)).*

We refer to the second ($\sigma^2 < \infty$, $\beta\mu > 1$) and third ($\sigma^2 = \infty$, $\beta\mu > 1$) cases as the finite variance, infinite excursion and infinite variance, infinite excursion cases respectively.

Definition 4.0.2. *(FVIE) The offspring distribution has mean* μ *satisfying* $1 < \mu^{-1} < \beta$ *and variance* $\sigma^2 < \infty$.

Definition 4.0.3. *(IVIE) The offspring distribution has mean* μ *satisfying* $1 < \mu^{-1} < \beta$ *and belongs to the domain of attraction of a stable law of index* $\alpha \in (1,2)$.

As for IVFE, in IVIE we let L vary slowly at ∞ such that (4.1) holds and $(a_n)_{n\geq 1}$ be a scaling sequence for the size-biased law such that for any $x > 0$, as $n \to \infty$ we have $\mathbf{P}(\xi^* \geq x a_n) \sim x^{-(\alpha-1)} n^{-1}$. It then follows that $a_n = n^{\frac{1}{\alpha-1}} \tilde{L}(n)$ for some slowly varying function $\tilde{L}$. In FVIE, $a_n = n$ will suffice.

In FVIE and IVIE the slowing is caused by excursions in deep branches because the walk is required to make long sequences of movements against the bias in order to escape. We shall see that only the depth $\mathcal{H}$ (and not the foliage) is important to the scaling. By comparison with the model in which we prune all of the branch except the unique self-avoiding path to the deepest point; we see that, by transience, the walk reaches the deepest point with positive probability and then takes a geometric number of short excursions with escape probability of the order $\beta^{-\mathcal{H}}$. In particular, this means that the expected time spent in a branch of height $\mathcal{H}$ will cluster around $\beta^{\mathcal{H}}$.

Intuitively, the main reason we observe different scalings in these two cases is due to the way the number of buds affects the height of the branch. The height of a GW-tree is approximately geometric; in particular, the tallest of n independent trees will typically be close to $\log(n)/\log(\mu^{-1})$. In FVIE the number of buds has finite mean therefore we see order n buds by level n hence tallest will have height close to $\log(n)/\log(\mu^{-1})$. In IVIE the number of buds has infinite mean but belongs to the domain of attraction of some stable law. In particular, the number of buds seen by level n is equal in distribution to the sum of n independent copies of $\xi^* - 1$ (which scales with a_n). It therefore follows that, in IVIE, the tallest tree up to level n will have height close to $\log(a_n)/\log(\mu^{-1})$. Since only the deepest trees are significant and the time spent in a large branch clusters around $\beta^{\mathcal{H}}$ we see that the natural scaling is $\beta^{\log(n)/\log(\mu^{-1})} = n^{1/\gamma}$ in FVIE and $\beta^{\log(a_n)/\log(\mu^{-1})} = a_n^{1/\gamma}$ in IVIE where we recall that the exponent γ is given in (1.2).

Since $\mathcal{H}$ is approximately geometric we have that $\beta^{\mathcal{H}}$ will not belong to the domain of attraction of any stable law. For this reason, as in [10], we only see convergence along specific increasing subsequences $n_l(t) := \lfloor t\mu^{-l} \rfloor$ for $t > 0$ in FVIE and $n_l(t)$ such that $a_{n_l(t)} \sim t\mu^{-l}$ for IVIE. Such a sequence exists for any $t > 0$ since, by choosing $n_l(t) := \sup\{m \geq 0 : a_m < t\mu^{-l}\}$, we have that $a_{n_l} < t\mu^{-l} \leq a_{n_l+1}$ and therefore

$$1 \geq \frac{a_{n_l}}{t\mu^{-l}} \geq \frac{a_{n_l}}{a_{n_l+1}} \to 1.$$

The main results for FVIE and IVIE are Theorems 4.2 and 4.3, which reflect slowing due to deep excursions.

Theorem 4.2. *In FVIE, for any $t > 0$ we have that as $l \to \infty$*

$$\frac{\Delta_{n_l(t)}}{n_l(t)^{\frac{1}{\gamma}}} \to R_t$$

in distribution under $\mathbb{P}$, where R_t is a random variable with an infinitely divisible law.

Theorem 4.3. *In IVIE, for any $t > 0$ we have that as $l \to \infty$*

$$\frac{\Delta_{n_l(t)}}{a_{n_l(t)}^{\frac{1}{\gamma}}} \to R_t$$

in distribution under $\mathbb{P}$, where R_t is a random variable with an infinitely divisible law.

We write r_n to be a_n in IVFE, $n^{1/\gamma}$ in FVIE and $a_n^{1/\gamma}$ in IVIE; then, letting $\overline{r}_n := \max\{m \geq 0 : r_m \leq n\}$ we will also prove Theorem 4.4. This shows that, although the laws of $X_n/\overline{r}_n$ do not converge in general (for FVIE and IVIE), the suitably scaled sequence is tight and we can determine the leading order polynomial exponent explicitly.

Theorem 4.4. *In IVFE, FVIE or IVIE we have that*

1. *The laws of $(\Delta_n/r_n)_{n\geq 0}$ under $\mathbb{P}$ are tight on $(0, \infty)$;*
2. *The laws of $(|X_n|/\overline{r}_n)_{n\geq 0}$ under $\mathbb{P}$ are tight on $(0, \infty)$.*

Moreover, in IVFE, FVIE and IVIE respectively, we have that $\mathbb{P}$-a.s.

$$\lim_{n\to\infty} \frac{\log|X_n|}{\log(n)} = \alpha - 1;$$
$$\lim_{n\to\infty} \frac{\log|X_n|}{\log(n)} = \gamma;$$
$$\lim_{n\to\infty} \frac{\log|X_n|}{\log(n)} = \gamma(\alpha - 1).$$

The proofs of Theorems 4.1, 4.2 and 4.3 follow a similar structure to the corresponding proof of [10] which, for the walk on the supercritical tree, only considers the case in which the variance of the offspring distribution is finite. However, for the latter reason, the proofs of Theorems 4.1 and 4.3 become more technical in some places, specifically with regards to the number of traps in a large branch. The proof can be broken down into a sequence of stages which investigate different aspects of the walk and the tree.

In all cases it will be important to decompose the time spent in large branches. In Section 4.1 we show a decomposition of the number of deep traps in any deep branch. This is only important for FVIE and IVIE since the depth of the branch plays a key role in decomposing the time spent in large branches. In Section 4.2 we determine conditions for labelling a branch as large in each of the regimes so that large branches are sufficiently far apart such that, with high probability, the embedded walk will not backtrack from one large branch to the previous one. In Section 4.3 we justify the choice of label by showing that time spent outside these large branches is negligible. From this we then have that Δ_n can be approximated by a sum of i.i.d. random variables whose distribution depends on n. In Section 4.4 we only consider IVFE and show that, under a suitable scaling, these variables converge in distribution which allows us to show the convergence of their sum. Similarly, in Section 4.5 we show that the random variables, suitably scaled, converge in distribution for FVIE and IVIE. We then show convergence of their sum in Section 4.6. In Section 4.7 we prove Theorem 4.4 which is standard following Theorems 4.1, 4.2 and 4.3.

Remark 4.0.4. *Theorem 4.1 was stated in [25] with the Skorohod J_1 topology instead of the M_1 topology. This is, in fact, false and we can only obtain the result under the (weaker) M_1 topology. To see that the result only holds under the M_1 topology notice that, since a single large branch is entered multiple times with positive probability, large jumps in the limit can arise from multiple large holding times for the discrete process. This phenomena does not allow for convergence in the J_1 topology. We will discuss this in greater detail in Section 4.3.*

4.1 The distribution of the number of traps in a large branch

In this section we show asymptotics for the tail probability of the height of a branch and use it to determine the distribution over the number of large traps in a large branch.

In the construction of the subcritical GW-tree conditioned to survive $\mathcal{T}$ described in Section 1.2.2, the special vertices form the infinite backbone $\mathcal{Y} = \{\rho_0, \rho_1, ...\}$ consisting of all vertices with an infinite line of descent where ρ_i is the vertex in generation i. Let $d_x := |c(x)|$ be the number of offspring of the vertex x. Each vertex ρ_i on the backbone is connected to buds $\rho_{i,j}$ for $j = 1, ..., d_{\rho_i} - 1$ (which are the normal

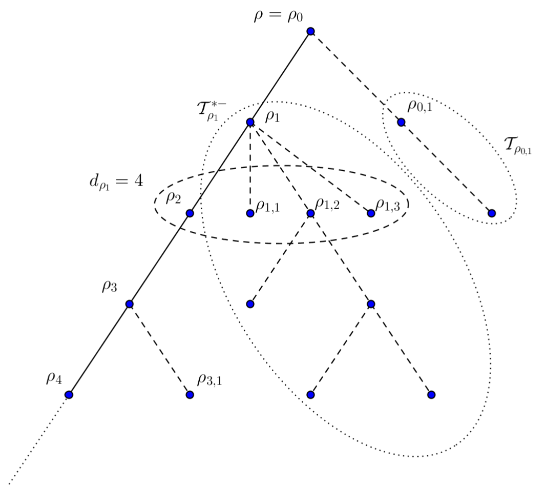

Figure 4.1: A sample section of a subcritical tree conditioned to survive with solid lines representing the backbone $\mathcal{Y}$ and dashed lines representing the dangling ends.

vertices that are offspring of special vertices in the construction). Each of these is then the root of an f-GW tree $\mathcal{T}_{\rho_{i,j}}$. We call each $\mathcal{T}_{\rho_{i,j}}$ a trap and the collection $\mathcal{T}^{*-}_{\rho_i}$ from a single backbone vertex (combined with the backbone vertex) a branch. Figure 4.1 shows an example of the first five generations of a tree $\mathcal{T}$. The solid line represents the backbone and the two dotted ellipses identify a sample branch and trap. The dashed ellipse indicates the children of ρ_1 which, since ρ_1 is on the backbone, have quantity distributed according to the size-biased law. It will be helpful throughout to work on a dummy branch which is equal in distribution to $\mathcal{T}^{*-}_{\rho_i}$ for any i thus we define the following random tree.

Definition 4.1.1. *(Dummy branch) Define $\mathcal{T}^{*-}$ to be a random tree rooted at ρ with first generation vertices $\rho_1, ..., \rho_{\xi^*-1}$ which are roots of independent f-GW-trees $(\mathcal{T}^f_i)_{i=1}^k$ where ξ^* is a size biased random variable independent of the rest of the tree. Define $\mathcal{T}^f$ to be a dummy f-GW-tree.*

The structure of the large traps will have an important role in determining the convergence of the scaled process. We will show that there is only a single deep trap at any backbone vertex when the offspring law has finite variance whereas, when the offspring law belongs to the domain of attraction of a stable law with index $\alpha < 2$, we have that the number of deep traps converges in distribution to a certain heavy tailed law.

Recall that $\mathcal{H}(\mathscr{T})$ denotes the height of a tree $\mathscr{T}$ rooted at ρ then by (2.9) we can define

$$s_m := \mathbf{P}(\mathcal{H}(\mathcal{T}^f) < m) = 1 - c_\mu \mu^m (1 + o(1)) \tag{4.2}$$

to be the probability that a given trap is of height at most $m-1$ (although in general we shall write s for convenience). Write

$$N(m) := \sum_{j=1}^{\xi^*-1} \mathbf{1}_{\{\mathcal{H}(\mathcal{T}^f_i) \geq m\}}$$

to be the number of traps of height at least m in the dummy branch then we are interested in the limit as $m \to \infty$ of

$$\mathbf{P}(N(m) = l | N(m) \geq 1) = \frac{\mathbf{P}(N(m) = l)}{\mathbf{P}(N(m) \geq 1)} \tag{4.3}$$

for $l \geq 1$. Recall that f is the p.g.f. of the offspring distribution, write $f^{(k)}$ for its k^{th} derivative then we have that

$$\mathbf{P}(N(m) = l) = \sum_{k=1}^{\infty} \mathbf{P}(\xi^* = k)\mathbf{P}(N(m) = l | \xi^* = k)$$

$$
\begin{aligned}
&= \sum_{k=l+1}^{\infty} \frac{kp_k}{\mu} s^{k-1-l}(1-s)^l \binom{k-1}{l} \\
&= \frac{(1-s)^l}{l!\mu} f^{(l+1)}(s). \qquad (4.4)
\end{aligned}
$$

In particular, we have that

$$\mathbf{P}(\mathcal{H}(\mathcal{T}^{*-}) > m) = \mathbf{P}(N(m) \geq 1) = 1 - f'(s)/\mu. \qquad (4.5)$$

Lemma 4.1.2 shows that, when $\sigma^2 < \infty$, with high probability there will only be a single deep trap in any deep branch.

Lemma 4.1.2. *When* $\sigma^2 < \infty$

$$\lim_{m\to\infty} \mathbf{P}(N(m) = 1|N(m) \geq 1) = 1.$$

Proof. Using (4.3) and (4.4) we have that

$$\mathbf{P}(N(m) = 1|N(m) \geq 1) \;=\; \frac{(1-s)f''(s)/\mu}{1-f'(s)/\mu} \;=\; \frac{\sum_{k=2}^{\infty} k(k-1)p_k s^{k-2}}{\sum_{k=2}^{\infty} kp_k \frac{1-s^{k-1}}{1-s}}. \qquad (4.6)$$

By monotonicity in s we have that

$$\lim_{s\to 1^-} \sum_{k=2}^{\infty} k(k-1)p_k s^{k-2} = \sum_{k=2}^{\infty} k(k-1)p_k$$

which is finite since $\sigma^2 < \infty$. Each summand in the denominator of (4.6) is increasing in s for $s \in (0,1)$ and by L'Hôpital's rule $1 - s^{k-1} \sim (k-1)(1-s)$ as $s \to 1^-$ therefore, by monotone convergence, the denominator in the final term of (4.6) converges to the same limit. $\square$

In order to determine the correct threshold for labelling a branch as large we will need to know the asymptotic form of $\mathbf{P}(N(m) \geq 1)$. Corollary 4.1.3 gives this for the finite variance case.

Corollary 4.1.3. *Suppose* $\sigma^2 < \infty$ *then*

$$\mathbf{P}(N(m) \geq 1) \;\sim\; c_\mu \mathbf{E}[\xi^* - 1]\mu^m \;=\; c_\mu \left(\frac{\sigma^2 + \mu^2}{\mu} - 1\right)\mu^m.$$

Proof. Let f_* denote the p.g.f. of ξ^* then $\mathbf{P}(N(m) \geq 1) = 1 - s^{-1}f_*(s)$. Since $\sigma^2 < \infty$ we have that $f_*'(s)$ exists and is continuous for $s \leq 1$ thus as $s \to 1^-$ we have that $f_*(1) - f_*(s) \sim (1-s)f_*'(1) = (1-s)\mathbf{E}[\xi^*]$. It therefore follows that

$$1 - s^{-1}f_*(s) \;=\; f_*(1) - f_*(s) - \frac{f_*(s)(1-s)}{s} \;\sim\; (1-s)(\mathbf{E}[\xi^*] - 1).$$

The result then follows by the definitions of c_μ (2.9) and s (4.2). $\square$

We now consider the case when $\sigma^2 = \infty$ but ξ belongs to the domain of attraction of a stable law of index $\alpha \in (1,2)$. Recall from Lemma 2.2.2 that as $s \to 1^-$

$$\mathbf{E}[s^\xi] - s^\mu \sim \frac{\Gamma(3-\alpha)}{\alpha(\alpha-1)}(1-s)^\alpha L((1-s)^{-1}).$$

It follows that there exists a function L_1 (which varies slowly as $s \to 1^-$) such that $\mathbf{E}[s^\xi] - s^\mu = (1-s)^\alpha L_1((1-s)^{-1})$ and

$$\lim_{s\to 1^-} \frac{L_1((1-s)^{-1})}{L((1-s)^{-1})} = \frac{\Gamma(3-\alpha)}{\alpha(\alpha-1)}. \tag{4.7}$$

Write $g(x) = x^\alpha L_1(x^{-1})$ so that $f(s) = s^\mu + g(1-s)$ then it follows that

$$f^{(l)}(s) = s^{\mu-l}(\mu)_l + (-1)^l g^{(l)}(1-s)$$

when this exists where $(\mu)_l := \prod_{j=0}^{l-1}(\mu - j)$ is the Pochhammer symbol. Write $L_2(x) := L_1(x^{-1})$ which is slowly varying at 0. It follows from [53, Theorem 2] that for all $l \in \mathbb{N}$ we have that $xg^{(l+1)}(x) \sim (\alpha - l)g^{(l)}(x)$ as $x \to 0$. Therefore, for any integer $l \geq 0$

$$\lim_{x\to 0^+} \frac{x^l g^{(l)}(x)}{g(x)} = (\alpha)_l. \tag{4.8}$$

Proposition 4.1.4 is the main result of this section and determines the limiting distribution of the number of traps of height at least m in a branch of height greater than m.

Proposition 4.1.4. *In IVIE, for $l \geq 1$ as $m \to \infty$*

$$\mathbf{P}(N(m) = l | N(m) \geq 1) \to \frac{1}{l!}\prod_{j=1}^{l} |\alpha - j|.$$

Proof. Recall that by (4.3) and (4.4) we want to determine the asymptotics of $1 - f'(s)/\mu$ and $(1-s)^l f^{(l+1)}(s)/(l!\mu)$ as $s \to 1^-$. We have that $1 - f'(s)/\mu = 1 - s^{\mu-1} + g'(1-s)/\mu$ and $g'(1-s) \sim \alpha(1-s)^{\alpha-1}L_2(1-s)$ as $s \to 1$. Since $\alpha < 2$, we have that $\lim_{s\to 1^-}(1-s^{\mu-1})(1-s)^{1-\alpha} = 0$ hence

$$1 - \frac{f'(s)}{\mu} \sim \frac{\alpha}{\mu}(1-s)^{\alpha-1}L_2(1-s). \tag{4.9}$$

For derivatives $l \geq 1$ we have that

$$\frac{(1-s)^l f^{(l+1)}(s)}{l!\mu} = \frac{(1-s)^l}{l!\mu}\left(s^{\mu-(l+1)}(\mu)_{l+1} + (-1)^{l+1} g^{(l+1)}(1-s)\right).$$

By (4.8) we have that $(1-s)^l g^{(l+1)}(1-s) \sim (\alpha)_{l+1}(1-s)^{\alpha-1}L_2(1-s)$. For $l \geq 1$ we have that $l+1-\alpha > 0$ hence

$$\frac{(1-s)^l f^{(l+1)}(s)}{l!\mu} \sim \frac{|(\alpha)_{l+1}|}{l!\mu}(1-s)^{\alpha-1}L_2(1-s). \tag{4.10}$$

Combining (4.3) with (4.9) and (4.10) gives the desired result. □

Proposition 4.1.4 will be useful for determining the number of large traps in a large branch but equally important is the asymptotic relation (4.9) which gives the tail behaviour of the height of a branch $\mathcal{T}^{*-}$. By the assumption on ξ that (4.1) holds we have that

$$\mathbf{P}(\xi^* \geq t) \sim \frac{2-\alpha}{\mu(\alpha-1)} t^{-(\alpha-1)}L(t) \tag{4.11}$$

as $t \to \infty$. Using (4.2), (4.5), (4.7), (4.9) and (4.11), we then have that

$$\mathbf{P}(\mathcal{H}(\mathcal{T}^{*-}) > m) \;\sim\; \frac{\Gamma(3-\alpha)c_\mu^{\alpha-1}}{\mu(\alpha-1)}\mu^{m(\alpha-1)}L(\mu^{-m}) \;\sim\; \Gamma(2-\alpha)c_\mu^{\alpha-1}\mathbf{P}(\xi^* \geq \mu^{-m}). \tag{4.12}$$

4.2 Large branches are far apart

In this section we introduce the conditions for a branch to be large. This will differ in each of the cases however, since many of the proofs will generalise to all three cases, we will use the same notation for some aspects.

In IVFE we will have that the slowing is caused by the large number of traps. In particular, we will be able to show that the time spent outside branches with a large number of buds is negligible.

Definition 4.2.1. *(IVFE large branch) For $\varepsilon \in (0,1)$ write*

$$l_{n,\varepsilon} := a_{\lfloor n^{1-\varepsilon} \rfloor} \qquad \text{and} \qquad l^+_{n,\varepsilon} := a_{\lfloor n^{1+\varepsilon} \rfloor}$$

then we have that $\mathbf{P}(\xi^* \geq l_{n,\varepsilon}) \sim n^{-(1-\varepsilon)}$. We will call a branch large if the number of buds is at least $l_{n,\varepsilon}$ and write $\mathcal{D}^{(n)} := \{x \in \mathcal{Y} : d_x > l_{n,\varepsilon}\}$ to be the collection of backbone vertices which are the roots of large branches.

In FVIE we will have that the slowing is caused by excursions into deep traps.

Definition 4.2.2. *(FVIE large branch) For $\varepsilon \in (0,1)$ write $C_{\mathcal{D}} := c_\mu \mathbf{E}[\xi^* - 1]$,*

$$h_{n,\varepsilon} := \left\lceil \frac{(1-\varepsilon)\log(n)}{\log(\mu^{-1})} \right\rceil \qquad \text{and} \qquad h^+_{n,\varepsilon} := \left\lceil \frac{(1+\varepsilon)\log(n)}{\log(\mu^{-1})} \right\rceil$$

then by Corollary 4.1.3 we have that

$$\mathbf{P}(\mathcal{H}(\mathcal{T}^{*-}) > h_{n,\varepsilon}) \;\sim\; C_{\mathcal{D}}\mu^{h_{n,\varepsilon}} \;\approx\; C_{\mathcal{D}}n^{-(1-\varepsilon)}.$$

We will call a branch large if there exists a trap within it of height at least $h_{n,\varepsilon}$ and write $\mathcal{D}^{(n)} := \{x \in \mathcal{Y} : \mathcal{H}(\mathcal{T}_x^{*-}) > h_{n,\varepsilon}\}$ to be the collection of backbone vertices which are the roots of large branches. By a large trap we mean any trap of height at least $h_{n,\varepsilon}$.

In IVIE we will have that the slowing is caused by a combination of the slowing effects of the other two cases. The height and number of buds in branches have a strong link which we show more precisely later; this allows us to label branches as large based on height which will be fundamental when decomposing the time spent in large branches.

Definition 4.2.3. *(IVIE large branch) For $\varepsilon \in (0,1)$ write*

$$h_{n,\varepsilon} := \left\lceil \frac{\log(a_{n^{1-\varepsilon}})}{\log(\mu^{-1})} \right\rceil \qquad \text{and} \qquad h^+_{n,\varepsilon} := \left\lceil \frac{\log(a_{n^{1+\varepsilon}})}{\log(\mu^{-1})} \right\rceil$$

then by (4.12), for $C_{\mathcal{D}} := \Gamma(2-\alpha)c_\mu^{\alpha-1}$, we have that

$$\mathbf{P}(\mathcal{H}(\mathcal{T}^{*-}) > h_{n,\varepsilon}) \;\sim\; C_{\mathcal{D}}\mathbf{P}(\xi^* \geq \mu^{-h_{n,\varepsilon}}) \;\approx\; C_{\mathcal{D}}n^{-(1-\varepsilon)}. \tag{4.13}$$

We will call a branch large if there exists a trap of height at least $h_{n,\varepsilon}$ and write $\mathcal{D}^{(n)} := \{x \in \mathcal{Y} : \mathcal{H}(\mathcal{T}_x^{*-}) > h_{n,\varepsilon}\}$ to be the collection of backbone vertices which are the roots of large branches. By a large trap we mean any trap of height at least $h_{n,\varepsilon}$.

We want to show that, asymptotically, the large branches are sufficiently far apart to ignore any correlation and therefore approximate Δ_n by the sum of i.i.d. random variables representing the time spent in a large branch. Much of this is very similar to [10] so we only give brief details.

Write $\mathcal{D}_m^{(n)} := \{x \in \mathcal{D}^{(n)} : 0 < |x| \leq m\}$ to be the large roots before level m then let $q_n := \mathbf{P}(\rho \in \mathcal{D}^{(n)})$ be the probability that a branch is large and write

$$A_1(n,T) := \left\{ \sup_{t \in [0,T]} \left| |\mathcal{D}^{(n)}_{\lfloor tn \rfloor}| - \lfloor tnq_n \rfloor \right| < n^{2\varepsilon/3} \right\} \tag{4.14}$$

to be the event that the number of large branches by level Tn does not differ too much from its expected value. Notice that in all three cases we have that q_n is of the order $n^{-(1-\varepsilon)}$ thus we expect to see $nq_n \approx Cn^\varepsilon$ large branches by level n.

Lemma 4.2.4. *In IVFE, FVIE and IVIE, for any $T > 0$*

$$\lim_{n\to\infty} \mathbf{P}\left(A_1(n,T)^c\right) = 0.$$

Proof. For each $n, m \in \mathbb{N}$ write

$$M_m^n := |\mathcal{D}_m^{(n)}| - mq_n \stackrel{\mathrm{d}}{=} \sum_{k=1}^{m} (B_k - q_n)$$

where B_k are independent Bernoulli random variables with success probability q_n. Then $\mathbf{E}[M_m^n] = 0$ and $\mathrm{Var}_{\mathbf{P}}(M_m^n) = mq_n(1-q_n)$ therefore by Kolmogorov's maximal inequality

$$\mathbf{P}\left(\max_{1\le m\le \lfloor nT\rfloor} |M_m^n| \ge n^{2\varepsilon/3} - 1\right) \;\le\; \frac{c\mathrm{Var}_{\mathbf{P}}\left(M_{\lfloor nT\rfloor}^n\right)}{n^{4\varepsilon/3}} \;\le\; \frac{cnTq_n}{n^{4\varepsilon/3}} \;\le\; CTn^{-\varepsilon/3}.$$

Since $|\lfloor nt\rfloor q_n - \lfloor ntq_n\rfloor| \le 1$ we have that

$$\begin{aligned}\sup_{t\in[0,T]} \left||\mathcal{D}_{\lfloor tn\rfloor}^{(n)}| - \lfloor tnq_n\rfloor\right| &\le \max_{1\le m\le\lfloor nT\rfloor} |M_m^n| + \sup_{0\le t\le T} |tnq_n - \lfloor nt\rfloor q_n| \\ &\le \max_{1\le m\le\lfloor nT\rfloor} |M_m^n| + 1\end{aligned}$$

which proves the statement. □

We want to show that all of the large branches are sufficiently far apart such that the walk does not backtrack from one to another. For $t > 0$ and $\kappa \in (0, 1-2\varepsilon)$ write

$$\mathcal{D}(n,t) := \left\{\min_{x,y\in\mathcal{D}_{\lfloor nt\rfloor}^{(n)}:\, x\neq y} d(x,y) > n^\kappa\right\} \cap \{\rho \notin \mathcal{D}^{(n)}\}$$

to be the event that all large branches up to level $\lfloor nt\rfloor$ are of distance at least n^κ apart and the root of the tree is not the root of a large branch. A union bound shows that $\mathbf{P}(\mathcal{D}(n,t)^c) \to 0$ as $n \to \infty$ uniformly over t in compact sets.

We want to show that, with high probability, once the walk reaches a large branch it never backtracks to the previous one. Recall that Y is the embedded walk of X on the backbone then write $\Delta_n^Y := \min\{m \ge 0 : Y_m = \rho_n\}$ to be the first hitting time of level n by Y. For $t > 0$ write

$$A_2^{(0)}(n,t) := \bigcap_{i=0}^{\lfloor nt\rfloor} \bigcap_{m\ge\Delta_{\rho_i}^Y} \{|Y_m| > i - \overline{C}\log(n)\}$$

to be the event that the walk never backtracks distance $\overline{C}\log(n)$ along the backbone. Recall that for $x \in \mathcal{T}$ we write $\tau_x^+ = \inf\{n > 0 : X_n = x\}$ to be the first return time of x. By comparison with a simple random walk on $\mathbb{Z}$, Lemma 2.3.2 shows that for $k \ge 1$ we have that the escape probability is $P_{\rho_k}\left(\tau_{\rho_{k-1}}^+ = \infty\right) = 1 - \beta^{-1}$ hence, using the Strong Markov property,

$$P_{\rho_m}\left(\tau^+_{\rho_0}<\infty\right)=\prod_{k=1}^{m}P_{\rho_k}\left(\tau^+_{\rho_{k-1}}<\infty\right)=\beta^{-m}.$$

Using a union bound we see that

$$\mathbb{P}(A_2^{(0)}(n,t)^c)\leq Cnt\beta^{-\overline{C}\log(n)}\to 0 \tag{4.15}$$

for $\overline{C}$ sufficiently large. Combining this with $\mathcal{D}(n,t)$ we have that with high probability the walk never backtracks from one large branch to a previous one.

4.3 Time is spent in large branches

In this section we show that the time spent up to time Δ_n outside large branches is negligible. Combined with Section 4.2 this allows us to approximate Δ_n by the sum of i.i.d. random variables.

Recall that Δ_n^Y is the first hitting time of ρ_n for the embedded walk Y and write

$$A_3(n):=\{\Delta_n^Y\leq C_1 n\}$$

to be the event that level n is reached by time C_1n by the walk on the backbone. Standard large deviation estimates yield that $\lim_{n\to\infty}\mathbb{P}(A_3(n)^c)=0$ for $C_1>(\beta+1)/(\beta-1)$.

For the remainder of this section we mainly consider the case in which ξ belongs to the domain of attraction of a stable law of index $\alpha\in(1,2)$. The case in which the offspring law has finite variance will proceed similarly however since the corresponding estimates are much simpler in this case we omit the proofs.

In IVIE and IVFE, for $t>0$, let the event that there are at most $\log(n)a_n$ buds by level $\lfloor nt\rfloor$ be

$$A_4(n,t):=\left\{\sum_{k=0}^{\lfloor nt\rfloor}(d_{\rho_k}-1)\leq\log(n)a_n\right\}.$$

The variables d_{ρ_k} are i.i.d. with the law of ξ^* therefore, since a_n/a_{nt} converges and the laws of

$$a_{\lfloor nt\rfloor}^{-1}\sum_{k=1}^{\lfloor nt\rfloor}(\xi_k^*-1)$$

converge to some stable law G^* where $\lim_{n\to\infty}\overline{G}^*(Ct^{\alpha-1}\log(n))=0$ uniformly over $t\leq T$, we have that $\lim_{n\to\infty}\mathbf{P}(A_4(n,t)^c)=0$.

In FVIE write

$$A_4(n,t) := \left\{\sum_{k=0}^{\lfloor nt\rfloor}(d_{\rho_k}-1) \leq \log(n)n\right\}$$

then Markov's inequality gives that $\lim_{n\to\infty}\mathbf{P}(A_4(n,t)^c) = 0$.

Write

$$A_5(n) := \left\{\max_{i,j} |\{k \leq \Delta_{\lfloor nt\rfloor} : X_{k-1} = \rho_i,\ X_k = \rho_{i,j}\}| \leq C_2\log(n)\right\} \tag{4.16}$$

to be the event that any trap is entered at most $C_2\log(n)$ times. By Lemma 2.3.3 the number of entrances into $\rho_{i,j}$ has the law of a geometric random variable of parameter $p = (\beta-1)/(2\beta-1)$ hence using a union bound we have that

$$\mathbb{P}(A_5(n,t)^c \cap A_4(n,t)) \;\leq\; \log(n)a_n\mathbb{P}\left(Geo(p) > C_2\log(n)\right) \;\leq\; \overline{L}(n)n^{\frac{1}{\alpha-1}+C_2\log(1-p)}$$

where $\overline{L}$ varies slowly at ∞. We therefore have that the final term converges to 0 for C_2 large and $\lim_{n\to\infty}\mathbb{P}(A_5(n,t)^c) = 0$.

Propositions 4.3.1, 4.3.2 and 4.3.3 show that any time spent outside large traps is negligible. In FVIE and IVIE we only consider the large traps in large branches. Recall that $\mathcal{D}^{(n)}$ is the set of roots of large branches and write

$$K(n) := \bigcup_{x\in\mathcal{D}^{(n)}}\{\mathcal{T}_y : y \in c(x)\setminus\{\rho_{|x|+1}\},\ \mathcal{H}(\mathcal{T}_y) \geq h_{n,\varepsilon}\}$$

to be the vertices in large traps. In IVFE we require the entire large branch and write

$$K(n) := \bigcup_{x\in\mathcal{D}^{(n)}}\mathcal{T}_x^{*-}$$

to be the vertices in large branches. In either case we write $\chi_{t,n} := |\{1 \leq i \leq \Delta_{\lfloor nt\rfloor} : X_{i-1}, X_i \in K(n)\}|$ to be the time spent up to $\Delta_{\lfloor nt\rfloor}$ in large traps.

Proposition 4.3.1. *In IVIE, for any $t,\epsilon > 0$ we have that as $n\to\infty$*

$$\mathbb{P}\left(\left|\frac{\Delta_{\lfloor nt\rfloor}-\chi_{t,n}}{a_n^{1/\gamma}}\right| \geq \epsilon\right) \to 0.$$

Proof. On $A_4(n,t)$ there are at most $a_n\log(n)$ traps by level $\lfloor nt\rfloor$. We can order these traps so write $T^{(l,k)}$ to be the duration of the k^{th} excursion into the l^{th} trap and $\rho(l)$ to be the root of this trap (that is, the unique bud in the trap). Here we consider an excursion to start from the bud and end at the last hitting time of the bud before returning to the backbone. Recall that on $A_3(n)$ the walk Y reaches level n by time

C_1n and on $A_5(n)$ no trap up to level n is entered more than $C_2\log(n)$ times. Using the estimates on A_3, A_4 and A_5 we have that

$$\mathbb{P}\left(\left|\frac{\Delta_{\lfloor nt\rfloor}-\chi_{t,n}}{a_n^{1/\gamma}}\right|\geq\epsilon\right)$$
$$\leq o(1)+\mathbb{P}\left(C_1nt+\sum_{l=0}^{a_n\log(n)}\sum_{k=1}^{C_2\log(nT)}\left(2+T^{(l,k)}\mathbf{1}_{\{\mathcal{H}(\mathcal{T}_{\rho(l)})<h_{n,\varepsilon}\}}\right)\geq\epsilon a_n^{\frac{1}{\gamma}}\right).$$

Since $a_n^{\frac{1}{\gamma}}\gg a_n\log(n)^2\gg n$, for n sufficiently large we have that, using Markov's inequality and (2.14) with $m=h_{n,\varepsilon}$, the second term can be bounded above by

$$2\epsilon^{-1}a_n^{-\frac{1}{\gamma}}\mathbb{E}\left[\sum_{l=0}^{a_n\log(n)}\sum_{k=1}^{C_2\log(nt)}T^{(l,k)}\mathbf{1}_{\{\mathcal{H}(\mathcal{T}_{\rho(l)})<h_{n,\varepsilon}\}}\right]\;\leq\;C_{t,\epsilon}\log(n)^2a_n^{1-\frac{1}{\gamma}}a_{n^{1-\varepsilon}}^{\frac{1}{\gamma}-1}.$$

Combining constants and slowly varying functions into a single function $L_{t,\epsilon}$ such that for any $\tilde{\varepsilon}>0$ we have that $L_{t,\epsilon}(n)\leq n^{\tilde{\varepsilon}}$ for n sufficiently large we then have that

$$\mathbb{P}\left(\left|\frac{\Delta_{\lfloor nt\rfloor}-\chi_{t,n}}{a_n^{1/\gamma}}\right|\geq\epsilon\right)\leq o(1)+L_{t,\epsilon}(n)n^{-\varepsilon\frac{\frac{1}{\gamma}-1}{\alpha-1}}$$

which converges to 0 since $\alpha,\frac{1}{\gamma}>1$. $\square$

Using A_3, A_5 and the form of A_4 for FVIE, the technique used to prove Proposition 4.3.1 extends straightforwardly to prove Proposition 4.3.2 therefore we omit the proof.

Proposition 4.3.2. *In FVIE, for any $t,\epsilon>0$ we have that as $n\to\infty$*

$$\mathbb{P}\left(\left|\frac{\Delta_{\lfloor nt\rfloor}-\chi_{t,n}}{n^{1/\gamma}}\right|\geq\epsilon\right)\to 0.$$

Similarly, we can show a corresponding result for IVFE.

Proposition 4.3.3. *In IVFE, for any $t,\epsilon>0$, as $n\to\infty$*

$$\mathbb{P}\left(\left|\frac{\Delta_{\lfloor nt\rfloor}-\chi_{t,n}}{a_n}\right|\geq\epsilon\right)\to 0.$$

Proof. We begin by bounding the total number of traps in small branches. As a result of the truncated moment function asymptotic (2.2) we have the following the truncated first moment asymptotic:

$$\mathbf{E}\left[\xi^*\mathbf{1}_{\{\xi^*\leq x\}}\right]\sim Cx^{2-\alpha}L(x)\tag{4.17}$$

as $x \to \infty$ for some constant C. Recall from Definition 4.2.1 that $l_{n,\varepsilon} \leq a_{n^{1-\varepsilon}}$. Let $c \in (0, 2-\alpha)$ then, by Markov's inequality and the truncated first moment asymptotic (4.17), for n large

$$\begin{aligned}\mathbb{P}\left(\sum_{k=0}^{\lfloor nt \rfloor}(d_{\rho_k}-1)\mathbf{1}_{\{d_{\rho_k}-1\leq l_{n,\varepsilon}\}} \geq n^{\frac{1-c\varepsilon}{\alpha-1}}\right) &\leq \frac{\mathbf{E}\left[\sum_{k=0}^{\lfloor nt \rfloor}(d_{\rho_k}-1)\mathbf{1}_{\{d_{\rho_k}-1\leq l_{n,\varepsilon}\}}\right]}{n^{\frac{1-c\varepsilon}{\alpha-1}}} \\ &\leq n^{-\frac{\varepsilon(2-\alpha-c)}{\alpha-1}}L_t(n)\end{aligned}$$

where $L_t(n)$ varies slowly at ∞. This converges to 0 as $n \to \infty$. We can order the traps in small branches and write $T^{(l,k)}$ to be the duration of the k^{th} excursion in the l^{th} trap not in a large branch where we consider an excursion to start and end at the backbone. Using A_3 and A_5 to bound the time taken by Y to reach level nt and the number of entrances into traps up to level nt we have that for n suitably large

$$\mathbb{P}\left(\left|\frac{\Delta_{\lfloor nt \rfloor}-\chi_{t,n}}{a_n}\right| \geq \epsilon\right) \leq o(1) + \mathbb{P}\left(\sum_{l=0}^{n^{\frac{1-c\varepsilon}{\alpha-1}}}\sum_{k=0}^{C_2\log(nT)} T^{(l,k)} \geq \frac{\epsilon}{2}a_n\right).$$

Using Markov's inequality on the final term yields

$$\mathbb{P}\left(\sum_{l=0}^{n^{\frac{1-c\varepsilon}{\alpha-1}}}\sum_{k=0}^{C_2\log(nT)} T^{(l,k)} \geq \frac{\epsilon}{2}a_n\right) \leq 2\epsilon^{-1}a_n^{-1}\mathbb{E}\left[\sum_{k=0}^{n^{\frac{1-c\varepsilon}{\alpha-1}}}\sum_{j=0}^{C_2\log(nT)} T^{(l,k)}\right] \leq n^{\frac{-c\varepsilon}{\alpha-1}}L_{T,\epsilon}(n)$$

for some $L_{t,\epsilon}$ varying slowly at ∞. This converges to 0 as $n \to \infty$ hence the result holds. $\square$

Recall that we write r_n to be a_n in IVFE, $n^{1/\gamma}$ in FVIE and $a_n^{1/\gamma}$ in IVIE. Since $\Delta_{\lfloor nt \rfloor} - \chi_{t,n}$ is non-negative and non-decreasing in t we have that

$$\sup_{0\leq t\leq T}|\Delta_{\lfloor nt \rfloor}-\chi_{t,n}| = |\Delta_{\lfloor nT \rfloor}-\chi_{T,n}|$$

therefore Corollary 4.3.4 follows from Propositions 4.3.1, 4.3.2 and 4.3.3.

Corollary 4.3.4. *In each of IVFE, FVIE and IVIE, for any $T > 0$*

$$\sup_{0\leq t\leq T}\frac{|\Delta_{\lfloor nt \rfloor}-\chi_{t,n}|}{r_n}$$

converges in $\mathbb{P}$-probability to 0.

Write χ_n^i to be the total time spent in large traps of the i^{th} large branch; that is

$$\chi_n^i := \left|\left\{m \geq 0 : X_{m-1}, X_m \in \left(\mathcal{T}^{*-}_{\rho_i^+} \cap K(n)\right)\right\}\right|$$

where ρ_i^+ is the element of $\mathcal{D}^{(n)}$ which is i^{th} closest to ρ. Notice that, whereas $\chi_{n,t}$ only accumulates time up to reaching $\rho_{\lfloor nt\rfloor}$, each χ_n^i may have contributions at arbitrarily large times. Recall that $A_2^{(0)}(n,t)$ is the event that the walk never backtracks distance $\overline{C}\log(n)$ along the backbone from a backbone vertex up to level $\lfloor nt\rfloor$. On $A_2^{(0)}(n,T)$ we therefore have that for all $t \leq T$

$$\sum_{i=1}^{|\mathcal{D}^{(n)}_{\lfloor nt-\overline{C}\log(n)\rfloor}|} \chi_n^i \;\leq\; \chi_{n,t} \;\leq\; \sum_{i=1}^{|\mathcal{D}^{(n)}_{\lfloor nt\rfloor}|} \chi_n^i \tag{4.18}$$

where, on $\mathcal{D}(n,t)$, the J_1 distance between the two sums in the above expression can be bounded above by $n^{-1}\overline{C}\log(n)$ plus a small error which comes from

$$\mathbf{P}\left(|\mathcal{D}^{(n)}_{\lfloor nT-\overline{C}\log(n)\rfloor}| \neq |\mathcal{D}^{(n)}_{\lfloor nT\rfloor}|\right)$$

which converges to 0 as $n\to\infty$. In particular, using that $A_2^{(0)}(n,T)$ and $\mathcal{D}(n,t)$ occur with high probability with the tightness result we prove in Section 4.7, in order to prove Theorems 4.1, 4.2 and 4.3 it will suffice to consider the time spent in large traps up to level $\lfloor nt\rfloor$ under the appropriate scaling.

Let $(X_n^{(i)})_{i\geq 1}$ be independent walks on the same tree $\mathcal{T}$ with the law of X_n and $(Y_n^{(i)})_{i\geq 1}$ the corresponding embedded walks. For $i\geq 1$ let $\tilde{\chi}_n^i$ be the time spent in the i^{th} large trap by $X_n^{(i)}$ and

$$\tilde{\chi}_{t,n} := \sum_{i=1}^{\lfloor ntq_n\rfloor} \tilde{\chi}_n^i.$$

The random variables $\tilde{\chi}_n^i$ are independent copies (under $\mathbb{P}$) of times spent in large branches; moreover, on $\mathcal{D}(n,t)$, we have that $\rho\notin\mathcal{D}^{(n)}$ therefore they are identically distributed. Let $\mathbb{P},\mathbb{E}$ extend to the enlarged space.

Recall that Λ is the set of strictly increasing continuous functions mapping $[0,T]$ onto itself and we consider the Skorohod J_1 metric defined in (2.1).

Lemma 4.3.5. *In each of IVFE, FVIE and IVIE,*

1. *as $n\to\infty$*

$$d_{J_1}\left(\left(\sum_{i=1}^{|\mathcal{D}^{(n)}_{\lfloor nt\rfloor}|}\frac{\chi_n^i}{r_n}\right)_{t\in[0,T]}, \left(\sum_{i=1}^{\lfloor tnq_n\rfloor}\frac{\chi_n^i}{r_n}\right)_{t\in[0,T]}\right)$$

   converges to 0 *in probability;*

2. *for any bounded $H : D([0,T],\mathbb{R})\to\mathbb{R}$ continuous with respect to the Skorohod*

J_1 topology, as $n \to \infty$

$$\left|\mathbb{E}\left[H\left(\left(\sum_{i=1}^{\lfloor tnq_n\rfloor}\frac{\chi_n^i}{r_n}\right)_{t\in[0,T]}\right)\right]-\mathbb{E}\left[H\left(\left(\sum_{i=1}^{\lfloor tnq_n\rfloor}\frac{\tilde{\chi}_n^i}{r_n}\right)_{t\in[0,T]}\right)\right]\right|\to 0.$$

Proof. By definition of d_{J_1}, the distance in statement 1 is equal to

$$\inf_{\lambda\in\Lambda}\sup_{t\in[0,T]}\left(\left|\sum_{i=1}^{|\mathcal{D}^{(n)}_{\lfloor nt\rfloor}|}\frac{\chi_n^i}{r_n}-\sum_{i=1}^{\lfloor\lambda(t)nq_n\rfloor}\frac{\chi_n^i}{r_n}\right|+|\lambda(t)-t|\right).$$

For $k=1,...,|\mathcal{D}^{(n)}_{\lfloor nT\rfloor}|\wedge\lfloor Tnq_n\rfloor$ let $t_k:=\inf\{t>0:|\mathcal{D}^{(n)}_{\lfloor nt\rfloor}|=k$ then define $\lambda_n(t_k)=k(nq_n)^{-1}$. Letting $\lambda_n(0)=0,\lambda_n(T)=T$ and $\lambda_n(s)$ be defined by the linear interpolation for the remaining points we have that $\lambda_n:[0,T]\to[0,T]$ is continuous and strictly increasing. It follows that

$$\sup_{t\in[0,T]}|t-\lambda_n(t)|\leq\sup_{t\in[0,T]}\left|t-\frac{|\mathcal{D}^{(n)}_{\lfloor nt\rfloor}|}{nq_n}\right|$$

which converges to 0 by Lemma 4.2.4 since $n^{2\varepsilon/3}(nq_n)^{-1}\to 0$. Moreover,

$$\sup_{t\in[0,T]}\left|\sum_{i=1}^{|\mathcal{D}^{(n)}_{\lfloor nt\rfloor}|}\frac{\chi_n^i}{r_n}-\sum_{i=1}^{\lfloor\lambda_n(t)nq_n\rfloor}\frac{\chi_n^i}{r_n}\right|\leq\sum_{i=|\mathcal{D}^{(n)}_{\lfloor nT\rfloor}|\wedge\lfloor Tnq_n\rfloor+1}^{|\mathcal{D}^{(n)}_{\lfloor nT\rfloor}|\vee\lfloor Tnq_n\rfloor}\frac{\chi_n^i}{r_n}.$$

It follows by independence of $|\mathcal{D}^{(n)}_{\lfloor nT\rfloor}|$ with χ_n^i, the tightness result Theorem 4.4 and Lemma 4.2.4 that this expression converges in probability to 0. This proves statement 1.

For $i\geq 1$ let

$$A_2^{(i)}(n,t):=\bigcap_{j=0}^{\lfloor nt\rfloor}\bigcap_{m\geq\Delta^{Y^{(i)}}_{\rho_j}}\{|Y_m^{(i)}|>j-\overline{C}\log(n)\}$$

be the analogue of $A_2^{(0)}(n,t)$ for the i^{th} copy and

$$\tilde{A}_2(n,t):=\mathcal{D}(n,t)\cap\{\rho\notin\mathcal{D}^{(n)}\}\cap\bigcap_{i=0}^{\lfloor ntq_n\rfloor}A_2^{(i)}(n,t)$$

be the event that ρ is not the root of a large branch, on each of the first $\lceil ntq_n\rceil$ copies the walk never backtracks distance $\overline{C}\log(n)$ and that large branches are of distance at least n^κ apart.

$$\mathbb{E}\left[H\left(\left(\sum_{i=1}^{\lfloor tnq_n\rfloor}\frac{\chi_n^i}{r_n}\right)_{t\in[0,T]}\right)\mathbf{1}_{\tilde{A}_2(n,T)}\right]=\mathbb{E}\left[H\left(\left(\sum_{i=1}^{\lfloor tnq_n\rfloor}\frac{\tilde{\chi}_n^i}{r_n}\right)_{t\in[0,T]}\right)\mathbf{1}_{\tilde{A}_2(n,T)}\right]$$

therefore

$$\left|\mathbb{E}\left[H\left(\left(\sum_{i=1}^{\lfloor tnq_n\rfloor}\frac{\chi_n^i}{r_n}\right)_{t\in[0,T]}\right)\right]-\mathbb{E}\left[H\left(\left(\sum_{i=1}^{\lfloor tnq_n\rfloor}\frac{\tilde{\chi}_n^i}{r_n}\right)_{t\in[0,T]}\right)\right]\right|$$
$$\leq ||H||_\infty\left(\lceil nTq_n\rceil\mathbb{P}(A_2^{(0)}(n,T)^c)+\mathbf{P}(\mathcal{D}(n,T)^c)+q_n\right)$$

which converges to 0 as $n\to\infty$ for $\overline{C}$ sufficiently large by using the same argument as (4.15) and that $\mathbf{P}(\mathcal{D}(n,T)^c)\to 0$. □

Using Corollary 4.3.4 and Lemma 4.3.5, in order to show the convergence of $\Delta_{\lfloor nt\rfloor}/r_n$, it suffices to show the convergence of the scaled sum of independent random variables $\tilde{\chi}_{t,n}/r_n$.

Remark 4.3.6. *Recall from Remark 4.0.4 that Theorem 4.1 was initially incorrectly believed to hold under the J_1 topology. The reason that we require the weaker topology is that, although the two sums in (4.18) are close in J_1 distance, it is not true that $\chi_{n,t}$ is close to either sum in J_1 distance. This inequality does, however, allow us to conclude that $\chi_{n,t}$ is close to the random sum in M_1 distance.*

4.4 Decomposing excursion times in dense branches

In this section we only consider IVFE. The main tool will be Proposition 2.3.1 which gives a set of conditions required to prove a random sum converges to a certain infinitely divisible law. In our case, the number of terms $n(t)$ will be the number of large branches up to level $\lfloor nt\rfloor$ and the summands $\{R_k\}_{k=1}^{n(t)}$ are independent copies of the time spent in a large branch.

Since we are now working with i.i.d. random variables we will simplify notation by considering the dummy branch $\mathcal{T}^{*-}$ defined in Definition 4.1.1 which has root ρ and first generation vertices $\rho_1,...,\rho_{\xi^*-1}$ which are roots of f-GW-trees $(\mathcal{T}_j^f)_{j=1}^{\xi^*-1}$. We then let $(W^j)_{j=1}^{\xi^*-1}$ have the multinomial distribution determined in Lemma 2.3.3; that is, W^j represents the number of excursions into the j^{th} trap of $\mathcal{T}^{*-}$. For the biased random walk X_n on $\mathcal{T}^{*-}$ started from ρ, let $T^{j,k}$ denote the duration of the k^{th} excursion in the j^{th} trap where we recall that in IVFE the excursion starts and ends at the root ρ. We then have that

$$\tilde{\chi}_n := \sum_{j=1}^{\xi^*-1}\sum_{k=1}^{W^j} T^{j,k}. \tag{4.19}$$

is equal in distribution under $\mathbb{P}(\cdot|\xi^* > l_{n,\varepsilon})$ to $\tilde{\chi}_n^i$ under $\mathbb{P}$ for any i.

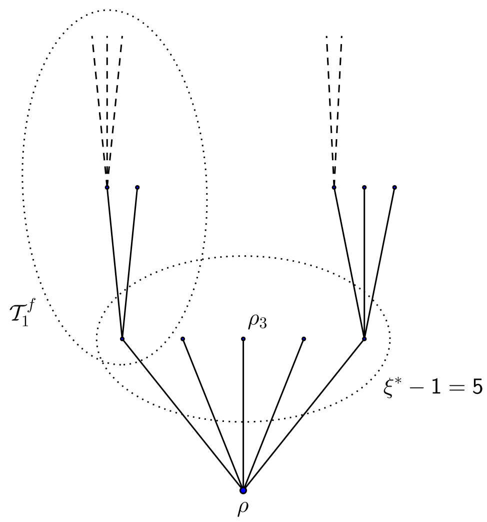

Figure 4.2: A dummy tree $\mathcal{T}^{*-}$ with five buds, each of which is the root of an independent, unconditioned subcritical GW-tree.

For $K \geq l_{n,\varepsilon} - l_{n,0}$ write $\overline{L}_K := l_{n,0} + K$ then denote $\mathbb{P}^K(\cdot) := \mathbb{P}\big(\cdot|\xi^* - 1 = \overline{L}_K\big)$ and $\mathbf{P}^K(\cdot) := \mathbf{P}\big(\cdot|\xi^* - 1 = \overline{L}_K\big)$. We now proceed to show that under $\mathbb{P}^K$

$$\zeta^{(n)} := \frac{1}{\xi^* - 1}\sum_{j=1}^{\xi^*-1}\sum_{k=1}^{W^j} T^{j,k} \tag{4.20}$$

converges in distribution to a random variable Z_∞ whose distribution does not depend on K.

We start by showing that the excursion times $T^{j,k}$ do not differ greatly from $E^{\mathcal{T}^{*-}}[T^{j,k}]$. In order to do this we require moment bounds on $T^{j,k}$ however, since $\mathbf{E}[\xi^2] = \infty$, we do not have finite variance of the excursion times and thus we require a more subtle treatment than standard large deviation estimates.

Recall that for a tree $\mathscr{T}$ we denote $Z_n^{\mathscr{T}}$ to be the size of the n^{th} generation. Excursion times are first return times τ_ρ^+ conditioned on the first step therefore pruning buds and using (2.7) we have that the expected excursion time in a trap $\mathcal{T}_j^f$ is

$$E_\rho^{\mathcal{T}^{*-}}[T^{j,k}] = E_\rho^{\mathcal{T}^{*-}}[\tau_\rho^+|X_1 = \rho_j] \leq 2\sum_{n=0}^{\infty} Z_n^{\mathcal{T}_j^f}\beta^n \leq 2\mathcal{H}(\mathcal{T}_j^f)\sup_n Z_n^{\mathcal{T}_j^f}\beta^n. \tag{4.21}$$

Using that $\mathbf{P}(\mathcal{H}(\mathcal{T}^f) \geq n) \sim c_\mu \mu^n$ (from (2.9)) we see that for n large there are no traps of height greater than $C\log(n)$ for some constant C thus for our purposes it will suffice to study $\sup_n Z_n^{\mathcal{T}^f}\beta^n$ which we understand by Lemma 2.4.2.

Lemma 4.4.1. *In IVFE, we can choose $\varepsilon > 0$ such that for any $t > 0$ there exists a constant C_t such that*

$$\sup_{K \geq -(a_n - l_{n,\varepsilon})} \mathbb{P}^K \left(\left| \frac{1}{\overline{L}_K} \sum_{j=1}^{\overline{L}_K} \sum_{k=1}^{W^j} (T^{j,k} - E^{\mathcal{T}^{*-}}[T^{j,1}]) \right| > t \right) \leq C_t n^{-2\varepsilon}.$$

Proof. Write $E_m := \bigcap_{j=1}^m \left\{ \mathcal{H}(\mathcal{T}_j^f) \leq C\log(m) \right\}$ to be the event that none of the first m trees have height greater than $C\log(m)$. Since $\mathbf{P}(\mathcal{H}(\mathcal{T}_j^f) \geq m) \sim c_\mu \mu^m$ we can choose $\tilde{C} > c_\mu$ such that

$$\mathbf{P}(E_m^c) \;\leq\; m\mathbf{P}(\mathcal{H}(\mathcal{T}_j^f) > C\log(m)) \;\leq\; \tilde{C} m \mu^{C\log(m)}.$$

Thus choosing $C > 1/\log(\mu^{-1})$ and $c = C\log(\mu^{-1}) - 1 > 0$ we have that $\mathbf{P}(E_m^c) \leq \tilde{C}m^{-c}$ for m sufficiently large. By Lemma 2.4.2 we have that $(Z_k\beta^k)^{1+\epsilon}$ is a supermartingale for $\epsilon > 0$ sufficiently small (where Z_k is the process associated to $\mathcal{T}^f$) therefore, by Doob's supermartingale inequality,

$$\mathbf{P}\left(\sup_{k \leq m} Z_k \beta^k \geq x \right) = \mathbf{P}\left(\sup_{k \leq m} (Z_k\beta^k)^{1+\epsilon} \geq x^{1+\epsilon} \right) \leq \mathbf{E}[Z_0^{1+\epsilon}]x^{-(1+\epsilon)}.$$

Using the expression (4.21) for the expected excursion time it follows that

$$\mathbf{P}\left(E^{\mathcal{T}_j^f}[T^{j,1}] > x | \mathcal{H}(\mathcal{T}_j^f) \leq C\log(m) \right) \leq C\log(m)^{1+\epsilon} x^{-(1+\epsilon)}. \tag{4.22}$$

In particular, for some slowly varying function $\overline{L}$

$$\mathbf{E}\left[E^{\mathcal{T}_j^f}[T^{j,1}]^2 \mathbf{1}_{\left\{ E^{\mathcal{T}_j^f}[T^{j,1}] \leq m \right\}} \middle| \mathcal{H}(\mathcal{T}_j^f) \leq C\log(m) \right] \leq C\overline{L}(m) m^{1-\epsilon}. \tag{4.23}$$

Let $\kappa = \epsilon/(2(1+\epsilon))$ then write $\overline{E}_m := E_m \cap \bigcap_{j=1}^m \{E^{\mathcal{T}_j^f}[T^{j,1}] \leq m^{1-\kappa}\}$ to be the event that no trap is of height greater than $C\log(m)$ and the expected excursion time in any trap is at most $m^{1-\kappa}$. For m sufficiently large, by (4.22) we have that

$$\begin{aligned} \mathbf{P}\left(\overline{E}_m^c \right) &\leq \mathbf{P}\left(\bigcup_{j=1}^m \{ E^{\mathcal{T}_j^f}[T^{j,1}] > m^{1-\kappa} \} \middle| \mathcal{H}(\mathcal{T}_j^f) \leq C\log(m) \; \forall j \leq m \right) + \mathbf{P}(E_m^c) \\ &\leq mC\log(m)^{1+\epsilon} m^{-(1-\kappa)(1+\epsilon)} + \tilde{C}m^{-c}. \end{aligned}$$

Write $\overline{\overline{E}}_m := \overline{E}_m \cap \bigcap_{j=1}^m \{W^j \leq C' \log(m)\}$ for $C' > (2\beta-1)/(\beta-1)$ to be the event that no trap is of height greater than $C\log(m)$, entered more than $C'\log(m)$ times or has expected excursion time greater than $m^{1-\kappa}$. Then, by a union bound and the geometric distribution of W^j from Lemma 2.3.3,

$$\begin{aligned}\mathbb{P}\left(\overline{\overline{E}}_m^c\right) &\leq \mathbf{P}(\overline{E}_m^c) + m\mathbb{P}(W^1 > C'\log(m)) \\ &\leq \tilde{C}\left(\log(m)^{1+\epsilon} m^{1-(1-\kappa)(1+\epsilon)} + m^{-c} + m^{1-C'\frac{\beta-1}{2\beta-1}}\right) \end{aligned} \tag{4.24}$$

for m large. We can choose $\varepsilon < \frac{1}{2}\min\left\{(1-\kappa)(1+\epsilon)-1,\ c,\ C'\frac{\beta-1}{2\beta-1}-1\right\}$ since $(1-\kappa)(1+\epsilon) > 1$. We then have that $\mathbb{P}\left(\overline{\overline{E}}_m^c\right) \leq \tilde{C}m^{-2\varepsilon}$ and

$$\begin{aligned}\mathbb{P}&\left(\left|\frac{1}{m}\sum_{j=1}^m\sum_{k=1}^{W^j}(T^{j,k} - E^{\mathcal{T}_j^f}[T_{j,k}])\right| > t\right) \\ &\leq \mathbb{E}\left[\frac{\sum_{j=1}^m C\log(m)\mathrm{Var}_{P^{\mathcal{T}_j^f}}((T^{j,1} - E^{\mathcal{T}_j^f}[T^{j,1}])\mathbf{1}_{\overline{\overline{E}}_m})}{(mt)^2}\right] + \mathbb{P}\left(\overline{\overline{E}}_m^c\right) \\ &\leq \frac{C\log(m)}{mt^2} m^{(1-\epsilon)}\overline{L}(m) + \tilde{C}m^{-2\varepsilon}\end{aligned}$$

for some slowly varying function $\overline{L}$. Here the first inequality comes from Chebyshev and the second holds due to (4.23). Since $\epsilon > 0$ we can choose $\varepsilon \in (0, \epsilon/2)$ then

$$\mathbb{P}\left(\left|\frac{1}{m}\sum_{j=1}^m\sum_{k=1}^{W^j}(T_{j,k} - E^{\mathcal{T}_j^f}[T_{j,k}])\right| > t\right) \leq C_t m^{-2\varepsilon}.$$

In particular, this holds for $m = \overline{L}_K \geq a_{n^{1-\varepsilon}}$ thus, since $\alpha < 2$,

$$\begin{aligned}\sup_{K \geq -(a_n - l_{n,\varepsilon})} \mathbb{P}^K\left(\left|\frac{1}{\overline{L}_K}\sum_{j=1}^{\overline{L}_K}\sum_{k=1}^{W^j}(T^{j,k} - E^{\mathcal{T}_j^f}[T^{j,1}])\right| > t\right) &\leq C_t \sup_{K \geq -(a_n - l_{n,\varepsilon})} \overline{L}_K^{-2\varepsilon} \\ &\leq C_t a_{n^{1-\varepsilon}}^{-2\varepsilon} \\ &= C_t n^{-2\varepsilon}\left(n^{\frac{2-\alpha}{\alpha-1}-\varepsilon}\tilde{L}(n^{1-\varepsilon})\right)^{-2\varepsilon}\end{aligned}$$

which is bounded above by $C_t n^{-2\varepsilon}$ for n large whenever $\varepsilon < (2-\alpha)/(\alpha-1)$. □

Using this we can now show that the average time spent in a trap converges to its expectation.

Lemma 4.4.2. *In IVFE, we can find $\varepsilon > 0$ such that for sufficiently large n we have* *that*

$$\sup_{K\geq -(a_n-l_{n,\varepsilon})}\mathbb{P}^K\left(\left|\frac{1}{\overline{L}_K}\sum_{j=1}^{\overline{L}_K}W^j(E^{\mathcal{T}_j^f}[T^{j,1}]-\mathbb{E}[T^{1,1}])\right|>t\right)\leq r(n)\left(n^{-\varepsilon}+\frac{C}{t}\right)$$

uniformly over $t\geq 0$ where $r(n)=o(1)$.

Proof. We continue using the notation defined in Lemma 4.4.1 and also define the event

$$E_m^j:=\{\mathcal{H}(\mathcal{T}_j^f)\leq \tilde{C}\log(m)\}\cap\{W^j\leq C\log(m)\}\cap\{E^{\mathcal{T}_j^f}[T^{j,1}]\leq m^{1-\kappa}\}$$

that the j^{th} trap is not tall, entered many times and that the expected excursion time in it is not large. Using the bound on $\mathbb{P}\left(\overline{\overline{E}}_m^c\right)$ we then have that

$$\begin{aligned}&\mathbb{P}\left(\left|\frac{1}{m}\sum_{j=1}^{m}W^j(E^{\mathcal{T}_j^f}[T^{j,1}]-\mathbb{E}[T^{1,1}])\right|>t\right)\\&\quad\leq\mathbb{E}\left[\mathbb{P}\left(\left|\frac{1}{m}\sum_{j=1}^{m}W^j(E^{\mathcal{T}_j^f}[T^{j,1}]\mathbf{1}_{E_m^j}-\mathbb{E}[T^{1,1}]\mathbf{1}_{E_m^j})\right|>t\middle|(W^j)_{j=1}^m\right)\right]+o(m^{-\varepsilon}).\end{aligned}$$

Since $\mathbb{E}[E^{\mathcal{T}_j^f}[T^{j,1}\mathbf{1}_{E_m^j}]]=\mathbb{E}[T^{j,1}\mathbf{1}_{E_m^j}]\neq\mathbb{E}[\mathbb{E}[T^{1,1}]\mathbf{1}_{E_m^j}]$ we have that the summand in the right hand side does not have zero mean thus we perform the splitting:

$$\begin{aligned}&\mathbb{E}\left[\mathbb{P}\left(\left|\frac{1}{m}\sum_{j=1}^{m}W^j(E^{\mathcal{T}_j^f}[T^{j,1}]\mathbf{1}_{E_m^j}-\mathbb{E}[T^{j,1}]\mathbf{1}_{E_m^j})\right|>t\middle|(W^j)_{j=1}^m\right)\right]\\&\quad\leq\mathbb{E}\left[\mathbb{P}\left(\left|\frac{1}{m}\sum_{j=1}^{m}W^j(E^{\mathcal{T}_j^f}[T^{j,1}]\mathbf{1}_{E_m^j}-\mathbb{E}[T^{j,1}\mathbf{1}_{E_m^j}])\right|>t/3\middle|(W^j)_{j=1}^m\right)\right]\\&\qquad+\mathbb{E}\left[\mathbb{P}\left(\left|\frac{1}{m}\sum_{j=1}^{m}W^j(\mathbb{E}[T^{j,1}\mathbf{1}_{E_m^j}]-\mathbb{E}[T^{j,1}\mathbf{1}_{E_m^j}]\mathbf{1}_{E_m^j})\right|>t/3\middle|(W^j)_{j=1}^m\right)\right]\\&\qquad+\mathbb{E}\left[\mathbb{P}\left(\left|\frac{1}{m}\sum_{j=1}^{m}W^j(\mathbb{E}[T^{j,1}]\mathbf{1}_{E_m^j}-\mathbb{E}[T^{j,1}\mathbf{1}_{E_m^j}]\mathbf{1}_{E_m^j})\right|>t/3\middle|(W^j)_{j=1}^m\right)\right].\end{aligned}$$

By Chebyshev's inequality and the tail bound $\mathbb{E}[E^{\mathcal{T}_j^f}[T^{j,1}]^2\mathbf{1}_{\{E_m^j\}}]\leq Cm^{1-\epsilon}L(m)$ from (4.23) we have that the first term is bounded above by

$$\frac{C\log(m)^2}{(mt/3)^2}\sum_{j=1}^{m}\mathrm{Var}_{\mathbb{P}}(E^{\mathcal{T}_j^f}[T^{j,1}]\mathbf{1}_{E_m^j})\leq C_t m^{-\epsilon}\overline{L}(m)$$

for some slowly varying function $\overline{L}$. The second term is equal to

$$\mathbb{E}\left[\mathbb{P}\left(\left|\frac{1}{m}\sum_{j=1}^{m} W^j\mathbb{E}[T^{1,1}\mathbf{1}_{E_m^j}]\mathbf{1}_{(E_m^j)^c}\right| > t/3\Big|(W^j)_{j=1}^m\right)\right] \leq \mathbf{P}\left(\bigcup_{j=1}^{m}(E_m^j)^c\right) = o(m^{-\varepsilon})$$

by (4.24). The final term can be written as

$$\begin{aligned}\mathbb{P}\left(\frac{1}{m}\sum_{j=1}^{m} W^j\mathbb{E}[T^{j,1}\mathbf{1}_{(E_m^j)^c}]\mathbf{1}_{E_m^j} > t/3\right) &\leq \frac{3}{mt}\sum_{j=1}^{m}\mathbb{E}[W^j]\mathbb{E}[T^{j,1}\mathbf{1}_{(E_m^j)^c}] \\ &= \frac{C}{t}\mathbb{E}[T^{1,1}\mathbf{1}_{(E_m^1)^c}]\end{aligned}$$

which converges to 0 as $m \to \infty$ by dominated convergence since, from (2.14), $\mathbb{E}[T^{1,1}] < \infty$. We therefore have that the statement holds by setting $m = \overline{L}_K$. ☐

Recall from (4.20) that, under $\mathbb{P}^K$, $\zeta^{(n)}$ is the average time spent in a trap of a branch with $\xi^* - 1 = \overline{L}_K$ buds. From Lemmas 4.4.1 and 4.4.2 we have that as $n \to \infty$

$$\sup_{K \geq -(a_n - l_{n,\varepsilon})} \mathbb{P}^K\left(\left|\zeta^{(n)} - \mathbb{E}[T^{1,1}]\sum_{j=1}^{\overline{L}_K}\frac{W^j}{\overline{L}_K}\right| > t\right) \to 0.$$

Using (2.7) we have that $\mathbb{E}[T^{1,1}] = 2/(1-\beta\mu)$. Write $\theta = (\beta-1)(1-\beta\mu)/(2\beta)$ and let $Z^\infty \sim \exp(\theta)$.

Corollary 4.4.3. *In IVFE, we can find $\varepsilon > 0$ such that for sufficiently large n we have that*

$$\sup_{K \geq -(a_n - l_{n,\varepsilon})}\left|\mathbb{P}^K\left(\zeta^{(n)} > t\right) - \mathbb{P}\left(Z^\infty > t\right)\right| \leq \tilde{r}(n)\left(n^{-\varepsilon} + \frac{C}{t}\right)$$

uniformly over $t \geq 0$ where $\tilde{r}(n) = o(1)$.

Proof. By Lemma 2.3.3 the sum of W^j have a geometric law. In particular,

$$\begin{aligned}&\left|\mathbb{P}(Z^\infty > t) - \mathbb{P}^K\left(\mathbb{E}[T^{1,1}]\sum_{j=1}^{\overline{L}_K}\frac{W^j}{\overline{L}_K} > t\right)\right| \\ &\quad = \left|e^{-\theta t} - \mathbb{P}^K\left(Geo\left(\frac{\beta-1}{(\overline{L}_K+1)\beta-1}\right) > \frac{\overline{L}_K t}{\mathbb{E}[T^{1,1}]}\right)\right| \\ &\quad = \left|e^{-\theta t} - \left(1 - \frac{\beta-1}{(\overline{L}_K+1)\beta-1}\right)^{\left\lceil\frac{\overline{L}_K t}{\mathbb{E}[T^{1,1}]}\right\rceil}\right| \\ &\quad = \left|e^{-\theta t} - e^{-\theta t\frac{\overline{L}_K\beta}{\overline{L}_K\beta+\beta-1}}\right| + o(\overline{L}_K^{-1})\end{aligned}$$

$$\leq Ce^{-\theta t}\overline{L}_K^{-1} + o(\overline{L}_K^{-1})$$

for some constant C independent of K. It therefore follows that the laws of $\zeta^{(n)}$ converge under $\mathbb{P}^K$ to an exponential law. In particular, using Lemmas 4.4.1 and 4.4.2 with the bound

$$\left|\mathbb{P}^K\left(\zeta^{(n)} > t\right) - \mathbb{P}\left(\mathbb{E}[T^{1,1}]\sum_{j=1}^{\overline{L}_K}\frac{W^j}{\overline{L}_K} > t\right)\right|$$
$$\leq \mathbb{P}^K\left(\left|\frac{1}{\overline{L}_K}\sum_{j=1}^{\overline{L}_K} W^j(E^{\mathcal{T}_j^f}[T^{j,1}] - \mathbb{E}[T^{1,1}])\right| > \epsilon\right) + \mathbb{P}(Z^\infty \in [t-\epsilon, t+\epsilon]) + O(\overline{L}_K^{-1})$$

with $\epsilon = r(n)^{1/2}t$, we have the result since $\overline{L}_K \geq l_{n,\varepsilon} \gg n^\varepsilon$. □

Corollary 4.4.4. *In IVFE, for any $\tau > 0$ fixed*

$$\lim_{n\to\infty}\sup_{C\geq 0}\sup_{K\geq -(a_n-l_{n,\varepsilon})}(C\vee 1)\left|\mathbb{E}\left[Z^\infty \mathbf{1}_{\{CZ^\infty\leq\tau\}}\right] - \mathbb{E}^K\left[\zeta^{(n)}\mathbf{1}_{\{C\zeta^{(n)}\leq\tau\}}\right]\right| = 0.$$

Proof. Let $\epsilon > 0$ then for $\overline{C} \in (1,\infty)$

$$\sup_{C\geq\overline{C}}\sup_{K\geq -(a_n-l_{n,\varepsilon})}(C\vee 1)\left|\mathbb{E}\left[Z^\infty \mathbf{1}_{\{CZ^\infty\leq\tau\}}\right] - \mathbb{E}^K\left[\zeta^{(n)}\mathbf{1}_{\{C\zeta^{(n)}\leq\tau\}}\right]\right|$$
$$\leq \sup_{C\geq\overline{C}}\sup_{K\geq -(a_n-l_{n,\varepsilon})}\mathbb{E}\left[CZ^\infty \mathbf{1}_{\{CZ^\infty\leq\tau\}}\right] + \mathbb{E}^K\left[C\zeta^{(n)}\mathbf{1}_{\{C\zeta^{(n)}\leq\tau\}}\right]$$
$$\leq \sup_{K\geq -(a_n-l_{n,\varepsilon})}\tau\left(\mathbb{P}(Z^\infty \leq \tau/\overline{C}) + \mathbb{P}^K(\zeta^{(n)} \leq \tau/\overline{C})\right). \tag{4.25}$$

By Corollary 4.4.3, for n sufficiently large, we can choose $\overline{C}$ sufficiently large such that (4.25) is bounded above by ϵ. For any K and n we have that $\mathbb{E}[Z^\infty] = \mathbb{E}^K[\zeta^{(n)}]$ therefore for $\epsilon \in (0,1)$

$$\sup_{C\leq\epsilon}\sup_{K\geq -(a_n-l_{n,\varepsilon})}\left|\mathbb{E}\left[Z^\infty \mathbf{1}_{\{Z^\infty\leq\tau/C\}}\right] - \mathbb{E}^K\left[\zeta^{(n)}\mathbf{1}_{\{\zeta^{(n)}\leq\tau/C\}}\right]\right|$$
$$= \sup_{C\leq\epsilon}\sup_{K\geq -(a_n-l_{n,\varepsilon})}\left|\mathbb{E}\left[Z^\infty \mathbf{1}_{\{Z^\infty>\tau/C\}}\right] - \mathbb{E}^K\left[\zeta^{(n)}\mathbf{1}_{\{\zeta^{(n)}>\tau/C\}}\right]\right|. \tag{4.26}$$

By Corollary 4.4.3 we can choose $\epsilon > 0$ sufficiently small such that (4.26) is bounded above by ϵ. It now follows that

$$\sup_{t\in[\epsilon,\overline{C}]}\sup_{K\geq -(a_n-l_{n,\varepsilon})}\left|\mathbb{E}\left[Z^\infty \mathbf{1}_{\{Z^\infty\leq t\}}\right] - \mathbb{E}^K\left[\zeta^{(n)}\mathbf{1}_{\{\zeta^{(n)}\leq t\}}\right]\right|$$
$$\leq \hat{C}\sup_{t\in[\epsilon,\overline{C}]}\sup_{K\geq -(a_n-l_{n,\varepsilon})}\left|\mathbb{P}^K\left(\zeta^{(n)} > t\right) - \mathbb{P}\left(Z^\infty > t\right)\right|$$

for some constant $\hat{C}$ thus the result follows by Corollary 4.4.3. $\square$

We write $\mathbb{P}^{>}(\cdot) := \mathbb{P}(\cdot|\xi^* > l_{n,\varepsilon})$ and $\mathbf{P}^{>}(\cdot) := \mathbf{P}(\cdot|\xi^* > l_{n,\varepsilon})$ to be the laws conditioned on the branch $\mathcal{T}^{*-}$ being large. From (4.19) we have that $\tilde{\chi}_n^i$ are independent copies of the time spent in a large branch. Define $\tilde{\chi}_n^{\infty} := (\xi^* - 1)Z^{\infty}$ where Z^{∞} is the exponential random variable used in Corollaries 4.4.3 and 4.4.4. Recall that $R_{d,\varsigma,\mathscr{L}}$ has the infinitely divisible law given by (2.6). Fix the sequence $(\lambda_n)_{n\geq 1}$ converging to some $\lambda > 0$ and denote $M_n^{\lambda} := \lfloor \lambda_n n^{\varepsilon} \rfloor$.

Proposition 4.4.5. *In IVFE, for any $\lambda > 0$, as $n \to \infty$*

$$\sum_{i=1}^{M_n^{\lambda}} \frac{\tilde{\chi}_n^i}{a_n} \xrightarrow{d} R_{d_\lambda,0,\mathscr{L}_\lambda}$$

where

$$d_\lambda = \int_0^\infty \frac{x}{1+x^2} \mathrm{d}\mathscr{L}_\lambda(x),$$
$$\mathscr{L}_\lambda(x) = \begin{cases} 0, & \text{if } x < 0, \\ -\lambda x^{-(\alpha-1)} \theta^{-(\alpha-1)} \Gamma(\alpha), & \text{if } x > 0. \end{cases}$$

Proof. Let $\epsilon > 0$ then by Markov's inequality

$$\begin{aligned} \mathbb{P}^{>}\left(\frac{\tilde{\chi}_n}{a_n} > \epsilon\right) &\leq \mathbb{P}^{>}(\xi^* - 1 \geq a_{n^{1-\varepsilon/2}}) + \mathbb{P}\left(\sum_{j=1}^{a_{n^{1-\varepsilon/2}}} \sum_{k=1}^{W^j} T^{j,k} \geq \epsilon a_n\right) \\ &\leq \frac{\mathbb{P}(\xi^* - 1 \geq a_{n^{1-\varepsilon/2}})}{\mathbb{P}(\xi^* - 1 \geq a_{n^{1-\varepsilon}})} + \frac{a_{n^{1-\varepsilon/2}}}{\epsilon a_n} \mathbb{E}[W^1]\mathbb{E}[T^{1,1}], \end{aligned}$$

which converges to 0 as $n \to \infty$. Thus, by Proposition 2.3.1, it suffices to show that

1.
$$\lim_{\tau \to 0^+} \limsup_{n \to \infty} M_n^{\lambda} \mathrm{Var}_{\mathbb{P}^{>}}\left(\frac{\tilde{\chi}_n}{a_n} \mathbf{1}_{\{\frac{\tilde{\chi}_n}{a_n} \leq \tau\}}\right) = 0,$$

2.
$$\mathscr{L}_\lambda(x) = \begin{cases} \lim_{n\to\infty} M_n^{\lambda} \mathbb{P}^{>}\left(\frac{\tilde{\chi}_n}{a_n} \leq x\right), & \text{if } x < 0, \\ -\lim_{n\to\infty} M_n^{\lambda} \mathbb{P}^{>}\left(\frac{\tilde{\chi}_n}{a_n} > x\right), & \text{if } x > 0, \end{cases}$$

3.
$$d_\lambda = \lim_{n\to\infty} M_n^{\lambda} \mathbb{E}^{>}\left[\frac{\tilde{\chi}_n}{a_n} \mathbf{1}_{\{\frac{\tilde{\chi}_n}{a_n} \leq \tau\}}\right] + \int_{|x|>\tau} \frac{x}{1+x^2} d\mathscr{L}_\lambda(x) - \int_{0<|x|\leq\tau} \frac{x^3}{1+x^2} d\mathscr{L}_\lambda(x)$$

where d_λ and $\mathscr{L}_\lambda$ are as stated above.

We start with the first condition and since $\lambda_n \to \lambda$ there exists a constant C such that

$$M_n^\lambda \mathrm{Var}_{\mathbb{P}^>}\left(\frac{\tilde{\chi}_n}{a_n}\mathbf{1}_{\{\frac{\tilde{\chi}_n}{a_n}\le\tau\}}\right) \le Cn^\varepsilon\mathbb{E}^>\left[\left(\frac{\tilde{\chi}_n}{a_n}\right)^2\mathbf{1}_{\{\frac{\tilde{\chi}_n}{a_n}\le\tau\}}\right] \tag{4.27}$$
$$\le Cn^\varepsilon\left(\tau^2\mathbf{P}^>(\xi^*-1\ge a_n)+\tau\mathbb{E}^>\left[\frac{\tilde{\chi}_n}{a_n}\mathbf{1}_{\{\xi^*-1<a_n\}}\right]\right).$$

By the definition of a_n we have that

$$\mathbf{P}^>(\xi^*-1\ge a_n) = \frac{\mathbf{P}(\xi^*\ge a_n)}{\mathbf{P}(\xi^*\ge a_{n^{1-\varepsilon}})} \sim n^{-\varepsilon}. \tag{4.28}$$

Conditional on the number of buds ξ^* we have that the number of excursions W^j into the j^{th} trap are independent from the excursion times $T^{j,k}$ and both the number of excursions and the excursion times have finite mean hence

$$\begin{aligned}\mathbb{E}^>\left[\frac{\tilde{\chi}_n}{a_n}\mathbf{1}_{\{\xi^*-1<a_n\}}\right] &= \sum_{r=a_{n^{1-\varepsilon}}}^{a_n-1}\frac{\mathbf{P}(\xi^*-1=r)}{\mathbf{P}(\xi^*-1\ge a_{n^{1-\varepsilon}})}\mathbb{E}\left[\sum_{j=1}^{r}\sum_{k=1}^{W^j}\frac{T^{j,k}}{a_n}\Big|\xi^*-1=r\right]\\ &\le \frac{\mathbb{E}[W^1]\mathbb{E}[T^{1,1}]}{\mathbf{P}(\xi^*-1\ge a_{n^{1-\varepsilon}})}\mathbf{E}\left[\frac{\xi^*-1}{a_n}\mathbf{1}_{\{\xi^*-1\le a_n\}}\right]\\ &\sim Cn^{-\varepsilon}\end{aligned}$$

where the asymptotic holds as $n\to\infty$ by (4.17). In particular, by combining this with (4.28) in (4.27) we have that $M_n^\lambda\mathrm{Var}_{\mathbb{P}^>}(\frac{\tilde{\chi}_n}{a_n}\mathbf{1}_{\{\frac{\tilde{\chi}_n}{a_n}\le\tau\}})\le C(\tau^2+\tau)$ for some constant C depending on λ hence, as $\tau\to 0^+$, we indeed have convergence to 0 and therefore the first condition holds.

We now move on to the Lévy spectral function $\mathscr{L}_\lambda$. Clearly, for $x<0$, we have that $\mathscr{L}_\lambda(x)=0$ since $\tilde{\chi}_n$ is a positive random variable. It therefore suffices to consider $x>0$. By Corollary 4.4.3 we have that the scaled time spent in a large trap $\zeta^{(n)}$ (from (4.20)) converges in distribution to an exponential random variable Z^∞ with parameter θ (which is independent of K) therefore, since $M_n^\lambda\sim\lambda n^\varepsilon$ and $\tilde{\chi}_n^\infty=(\xi^*-1)Z^\infty$, we have that

$$\begin{aligned}M_n^\lambda\mathbb{P}^>\left(\frac{\tilde{\chi}_n^\infty}{a_n}>x\right) &\sim \lambda n^\varepsilon\mathbb{P}^>\left((\xi^*-1)Z^\infty>xa_n\right)\\ &\sim \frac{\lambda}{\mathbf{P}(\xi^*-1\ge a_n)}\sum_{K\ge l_{n,\varepsilon}-l_{n,0}}\mathbf{P}(\xi^*-1=\overline{L}_K)\mathbb{P}(\overline{L}_KZ^\infty>xa_n)\\ &= \lambda\frac{\mathbb{P}((\xi^*-1)Z^\infty\ge xa_n)}{\mathbf{P}(\xi^*-1\ge a_n)}-\sum_{j=0}^{l_{n,\varepsilon}-1}\frac{\lambda\mathbf{P}(\xi^*-1=j)\mathbb{P}(jZ^\infty>xa_n)}{\mathbf{P}(\xi^*-1\ge a_n)}\\ &\sim \lambda\theta^{-(\alpha-1)}\Gamma(\alpha)x^{-(\alpha-1)}.\end{aligned}$$

Where the final asymptotic holds by Lemma 2.2.1 and because

$$\sum_{j=0}^{l_{n,\varepsilon}-1} \frac{\lambda \mathbf{P}(\xi^* - 1 = j)\mathbb{P}(jZ^\infty > xa_n)}{\mathbf{P}(\xi^* - 1 \geq a_n)} \leq \lambda \frac{\mathbb{P}(Z^\infty > xa_n/a_{n^{1-\varepsilon}})}{\mathbf{P}(\xi^* - 1 \geq a_n)} = \lambda \frac{e^{-\theta x \frac{a_n}{l_{n,\varepsilon}}}}{\mathbf{P}(\xi^* - 1 \geq a_n)}$$

which converges to 0 as $n \to \infty$ since $l_{n,\varepsilon} = a_{\lfloor n^{1-\varepsilon} \rfloor}$ (and therefore $a_n/l_{n,\varepsilon} \gg n^\varepsilon$). It now suffices to show that $n^\varepsilon \left(\mathbb{P}^> \left(\frac{\tilde{\chi}_n^\infty}{a_n} > x\right) - \mathbb{P}^> \left(\frac{\tilde{\chi}_n}{a_n} > x\right)\right)$ converges to 0 as $n \to \infty$. To do this we condition on the number of buds:

$$\mathbb{P}^> \left(\frac{\tilde{\chi}_n^\infty}{a_n} > x\right) - \mathbb{P}^> \left(\frac{\tilde{\chi}_n}{a_n} > x\right) = \sum_{K \geq l_{n,\varepsilon} - l_{n,0}} \mathbf{P}^>(\xi^* - 1 = \overline{L}_K) \left(\mathbb{P}\left(\frac{\overline{L}_K Z^\infty}{a_n} > x\right) - \mathbb{P}^K \left(\frac{\overline{L}_K \zeta^{(n)}}{a_n} > x\right)\right).$$

We consider positive and negative K separately. For $K \geq 0$ we have that

$$\begin{aligned}\sum_{K=0}^{\infty} n^\varepsilon \mathbf{P}^>(\xi^* - 1 = \overline{L}_K) &\left|\mathbb{P}\left(\frac{\overline{L}_K Z^\infty}{a_n} > x\right) - \mathbb{P}^K \left(\frac{\overline{L}_K \zeta^{(n)}}{a_n} > x\right)\right| \\ &\leq n^\varepsilon \mathbf{P}^>(\xi^* - 1 \geq a_n) \sup_{c \leq 1,\ K \geq 0} \left|\mathbb{P}^K(Z^\infty > cx) - \mathbb{P}^K(\zeta^{(n)} > cx)\right|. \qquad (4.29)\end{aligned}$$

By (4.28) $n^\varepsilon \mathbf{P}^>(\xi^* - 1 \geq a_n)$ converges as $n \to \infty$ hence, using Corollary 4.4.3, the right hand side in (4.29) converges to 0. For $K \leq 0$, by Corollary 4.4.3 we have that

$$\begin{aligned}\sum_{K=-\infty}^{0} \mathbf{1}_{\{K \geq l_{n,\varepsilon} - l_{n,0}\}} n^\varepsilon \mathbf{P}^>(\xi^* - 1 = \overline{L}_K) &\left|\mathbb{P}\left(\frac{\overline{L}_K Z^\infty}{a_n} > x\right) - \mathbb{P}^K \left(\frac{\overline{L}_K \zeta^{(n)}}{a_n} > x\right)\right| \\ &\leq n^\varepsilon \sum_{K=-\infty}^{0} \mathbf{1}_{\{K \geq l_{n,\varepsilon} - l_{n,0}\}} \mathbf{P}^>(\xi^* - 1 = \overline{L}_K) \tilde{r}(n) \left(n^{-\varepsilon} + \frac{C_x \overline{L}_K}{a_n}\right) \\ &\leq o(1) + \frac{C_x \tilde{r}(n) n^\varepsilon}{a_n} \sum_{K=-\infty}^{0} \mathbf{1}_{\{K \geq l_{n,\varepsilon} - l_{n,0}\}} \frac{\mathbf{P}(\xi^* - 1 = \overline{L}_K)}{\mathbf{P}(\xi^* - 1 \geq l_{n,\varepsilon})} \overline{L}_K.\end{aligned}$$

For some constant C we have that $\mathbf{P}(\xi^* - 1 \geq l_{n,\varepsilon}) \sim Cn^{-(1-\varepsilon)}$ thus by (4.17)

$$\begin{aligned}\frac{C_x \tilde{r}(n) n^\varepsilon}{a_n} \sum_{K=-\infty}^{0} \mathbf{1}_{\{K \geq l_{n,\varepsilon} - l_{n,0}\}} \frac{\mathbf{P}(\xi^* - 1 = \overline{L}_K)}{\mathbf{P}(\xi^* - 1 \geq l_{n,\varepsilon})} \overline{L}_K &\leq C_x \tilde{r}(n) n \mathbb{E}\left[\frac{\xi^* - 1}{a_n} \mathbf{1}_{\{\xi^* - 1 \leq a_n\}}\right] \\ &\sim C_x \tilde{r}(n).\end{aligned}$$

In particular, since $\tilde{r}(n) = o(1)$, we indeed have that this converges to zero and thus we have the required convergence for $\mathscr{L}_\lambda$.

Finally, we consider the drift term d_λ. Since $\int_{0<x\leq\tau} x \mathrm{d}\mathscr{L}_\lambda(x) < \infty$ we have that

$$d_\lambda = \lim_{n\to\infty} M_n^\lambda \mathbb{E}^> \left[\frac{\tilde{X}_n}{a_n}\mathbf{1}_{\{\frac{\tilde{X}_n}{a_n}\leq \tau\}}\right] + \int_0^\infty \frac{x}{1+x^2} d\mathscr{L}_\lambda(x) - \int_0^\tau x \mathrm{d}\mathscr{L}_\lambda(x).$$

We want to show that $d_\lambda = \int_0^\infty \frac{x}{1+x^2}\mathrm{d}\mathscr{L}_\lambda(x)$ thus we need to show that the other terms cancel. By definition of $\mathbb{P}^>$ we have that

$$\mathbb{E}^> \left[\frac{\tilde{X}_n^\infty}{a_n}\mathbf{1}_{\{\frac{\tilde{X}_n^\infty}{a_n}\leq \tau\}}\right] = \frac{1}{\mathbf{P}(\xi^*-1\geq l_{n,\varepsilon})}\mathbb{E}\left[\frac{(\xi^*-1)Z^\infty}{a_n}\mathbf{1}_{\{\frac{(\xi^*-1)Z^\infty}{a_n}\leq\tau\}\cap\{\xi^*>l_{n,\varepsilon}\}}\right].$$

By Lemma 2.2.1, $(\xi^*-1)Z^\infty$ belongs to the domain of attraction of a stable law of index $\alpha-1$ and satisfies the scaling properties of ξ^* (up to a constant factor). Therefore, using that $a_n \gg l_{n,\varepsilon}$, we have that

$$\begin{aligned} M_n^\lambda \mathbb{E}^> \left[\frac{\tilde{X}_n^\infty}{a_n}\mathbf{1}_{\{\frac{\tilde{X}_n^\infty}{a_n}\leq \tau\}}\right] &\sim \frac{\lambda n^\varepsilon}{\mathbf{P}(\xi^*-1\geq l_{n,\varepsilon})}\mathbb{E}\left[\frac{(\xi^*-1)Z^\infty}{a_n}\mathbf{1}_{\{\frac{(\xi^*-1)Z^\infty}{a_n}\leq\tau\}}\right] \\ &\sim \frac{\alpha-1}{2-\alpha}\tau^{2-\alpha}\lambda\theta^{-(\alpha-1)}\Gamma(\alpha). \end{aligned}$$

Using the form of the Lévy spectral function we have that

$$\int_0^\tau x\mathrm{d}\mathscr{L}_\lambda(x) \;=\; \lambda\theta^{-(\alpha-1)}\Gamma(\alpha)\int_{\tau^{-(\alpha-1)}}^\infty x^{-\frac{1}{\alpha-1}}\mathrm{d}x \;=\; \frac{\alpha-1}{2-\alpha}\tau^{2-\alpha}\lambda\theta^{-(\alpha-1)}\Gamma(\alpha)$$

thus it remains to show that

$$n^\varepsilon\left(\mathbb{E}^> \left[\frac{\tilde{X}_n^\infty}{a_n}\mathbf{1}_{\{\frac{\tilde{X}_n^\infty}{a_n}\leq \tau\}}\right] - \mathbb{E}^> \left[\frac{\tilde{X}_n}{a_n}\mathbf{1}_{\{\frac{\tilde{X}_n}{a_n}\leq \tau\}}\right]\right) \to 0.$$

Similarly to the previous parts we condition on $\xi^*-1 = \overline{L}_K$ and consider the sums over K positive and negative separately. For $K\leq 0$

$$n^\varepsilon \sum_{K\leq 0}\mathbf{P}^>(\xi^*-1=\overline{L}_K)\left|\mathbb{E}\left[\frac{\overline{L}_K Z^\infty}{a_n}\mathbf{1}_{\left\{\frac{\overline{L}_K Z^\infty}{a_n}\leq\tau\right\}}\right] - \mathbb{E}^K\left[\frac{\overline{L}_K\zeta^{(n)}}{a_n}\mathbf{1}_{\left\{\frac{\overline{L}_K\zeta^{(n)}}{a_n}\leq\tau\right\}}\right]\right|$$

is bounded above by

$$\frac{n^\varepsilon}{\mathbf{P}(\xi^*\geq l_{n,\varepsilon})}\mathbb{E}\left[\frac{\xi^*-1}{a_n}\mathbf{1}_{\{\xi^*\leq a_n\}}\right]\sup_{K\leq 0}\left|\mathbb{E}\left[Z^\infty\mathbf{1}_{\left\{Z^\infty\leq\tau\frac{a_n}{\overline{L}_K}\right\}}\right] - \mathbb{E}^K\left[\zeta^{(n)}\mathbf{1}_{\left\{\zeta^{(n)}\leq\tau\frac{a_n}{\overline{L}_K}\right\}}\right]\right|.$$

By definition of $l_{n,\varepsilon}$ and properties of stable laws $n^\varepsilon\mathbb{E}\left[(\xi^*-1)/a_n\mathbf{1}_{\{\xi^*\leq a_n\}}\right]/\mathbf{P}(\xi^*\geq l_{n,\varepsilon})$ converges to some constant as $n\to\infty$. By Corollary 4.4.4 we therefore have that this converges to 0. Similarly for $K \geq 0$ we have that

$$
\begin{aligned}
n^{\varepsilon} \sum_{K\geq 0} \mathbf{P}^{>}(\xi^* - 1 = \overline{L}_K) & \left| \mathbb{E}\left[\frac{\overline{L}_K Z^{\infty}}{a_n}\mathbf{1}_{\left\{\frac{\overline{L}_K Z^{\infty}}{a_n} \leq \tau\right\}}\right] - \mathbb{E}^K\left[\frac{\overline{L}_K \zeta^{(n)}}{a_n}\mathbf{1}_{\left\{\frac{\overline{L}_K \zeta^{(n)}}{a_n} \leq \tau\right\}}\right]\right| \\
& \leq \frac{n^{\varepsilon}\mathbf{P}(\xi^* \geq l_{n,0})}{\mathbf{P}(\xi^* \geq l_{n,\varepsilon})} \sup_{K\geq 0} \frac{\overline{L}_K}{a_n} \left| \mathbb{E}\left[Z^{\infty}\mathbf{1}_{\left\{Z^{\infty} \leq \tau\frac{a_n}{\overline{L}_K}\right\}}\right] - \mathbb{E}^K\left[\zeta^{(n)}\mathbf{1}_{\left\{\zeta^{(n)} \leq \tau\frac{a_n}{\overline{L}_K}\right\}}\right]\right|.
\end{aligned}
$$

We have that $n^{\varepsilon}\mathbf{P}(\xi^* \geq l_{n,0})/\mathbf{P}(\xi^* \geq l_{n,\varepsilon})$ converges to some constant as $n \to \infty$. The result then follows by Corollary 4.4.4. □

This shows the convergence result of Theorem 4.1 in the sense of one dimensional distributions. Convergence of final dimensional distributions follows from the fact that we consider an i.i.d. sum. In Section 4.7 we prove a tightness result which concludes the proof.

4.5 Decomposing excursion times in deep branches

In this section we decompose the time spent in deep branches. In FVIE this will be very similar to the decomposition used in [10] and we will not consider the argument in great detail. However, the decomposition required in IVIE requires greater delicacy.

In Lemma 4.5.1 and Proposition 4.5.2 we consider a construction of a GW-tree conditioned on its height from [38] to show that the time spent in deep traps essentially consists of some geometric number of excursions from the deepest point (apex) in the trap to itself. That is, as in [10], excursions which do not reach the apex are negligible as is the time taken for the walk to reach the apex from the root of the trap and the time taken to return to the root from the apex when this happens before returning to the apex.

Following this we show that, conditional on the exact height of the branch $\overline{H}$, the time spent in the branch scaled by $\beta^{\overline{H}}$ converges in distribution along the given subsequences. In Lemma 4.5.5 we determine an important asymptotic relation for the distribution over the number of buds conditional on the height of the branch. In Lemmas 4.5.6-4.5.9 we provide various bounds which allow us, in Proposition 4.5.10, to show that the excursion time in a large branch is close to the random variable Z_{∞}^{n} (defined in (4.48)) which removes some of the dependency on n.

The main result of the section is Proposition 4.5.14 which shows that the scaled time spent in a large branch converges in distribution along the given subsequences. As a prelude to this we prove Lemmas 4.5.11-4.5.13 which show that we can reintroduce small traps into the branch and that the height of a trap is sufficiently close to a geometric random variable. We then conclude the section by showing that the scaled excursion times can be dominated by some random variable with a certain moment property which will be important in Section 4.6.

Recall that $\mathcal{T}^f$ is an f-GW-tree and $\mathcal{H}(\mathcal{T}^f)$ is its height then, following notation of [10], we denote by $(\phi_{n+1}, \psi_{n+1})_{n\geq 0}$ a sequence of i.i.d. pairs with joint law

$$\begin{aligned}\mathbf{P}(\phi_{n+1} = j, \psi_{n+1} = k) &:= \\ &\frac{\mathbf{P}(\xi = k)\mathbf{P}(\mathcal{H}(\mathcal{T}^f) \leq n-1)^{j-1}\mathbf{P}(\mathcal{H}(\mathcal{T}^f) = n)\mathbf{P}(\mathcal{H}(\mathcal{T}^f) \leq n)^{k-j}}{\mathbf{P}(\mathcal{H}(\mathcal{T}^f) = n+1)}\end{aligned} \tag{4.30}$$

for $k = 1, 2, ...$ and $j = 1, ..., k$. Under this law ψ_{n+1} has the law of the degree of the root of a GW-tree conditioned to be of height $n+1$ and ϕ_{n+1} has the law over the first bud to give rise onto a tree of height exactly n. We then construct a sequence of trees recursively as follows: Set $\mathcal{T}_0^{\prec} = \{\delta\}$ then

1. Let the first generation of $\mathcal{T}_{n+1}^{\prec}$ be of size ψ_{n+1}.

2. Attach $\mathcal{T}_n^{\prec}$ to the ϕ_{n+1}^{th} first generation vertex of $\mathcal{T}_{n+1}^{\prec}$.

3. Attach f-GW-trees conditioned to have height at most $n-1$ to the first $\phi_{n+1}-1$ vertices of the first generation of $\mathcal{T}_{n+1}^{\prec}$.

4. Attach f-GW-trees conditioned to have height at most n to the remaining $\psi_{n+1}-\phi_{n+1}$ first generation vertices of $\mathcal{T}_{n+1}^{\prec}$.

Under this construction $\mathcal{T}_{n+1}^{\prec}$ has the distribution of an f-GW-tree conditioned to have height exactly $n+1$. Write $\delta_0 = \delta$ to be the apex of the tree and for $n = 1, 2, ...$ write δ_n to be the ancestor of δ of distance n. The sequence $\delta_0, \delta_1, ...$ form a 'spine' from the apex to the root of the tree. We denote $\mathcal{T}^{\prec}$ to be the tree asymptotically attained. By a subtrap of $\mathcal{T}^{\prec}$ we mean some vertex x on the spine together with a child y off the spine and all of the descendants of y. This is itself a tree with root x and we write $\mathcal{S}_x$ to be the collection of subtraps rooted at x. Figure 4.3 shows a construction of $\mathcal{T}_4^{\prec}$ where the solid line represents the spine and the dashed lines represent subtraps.

We denote by $\mathcal{S}^{n,j,1}$ the j^{th} subtrap conditioned to have height at most $n-1$ attached to δ_n and $\mathcal{S}^{n,j,2}$ to be the j^{th} subtrap conditioned to have height at most n attached to δ_n. Recall that $d(x, y)$ denotes the graph distance between vertices x, y then for $k = 1, 2$ let

$$\Pi^{n,j,k} := 2 \sum_{x\in\mathcal{S}^{n,j,k}\setminus\{\delta_n\}} \beta^{d(x,\delta_n)}$$

denote the weight of $\mathcal{S}^{n,j,k}$ under the invariant measure associated to the conductance model with conductances β^{i+1} between levels $i, i+1$ and the roots of $\mathcal{S}^{n,j,k}$ (spinal

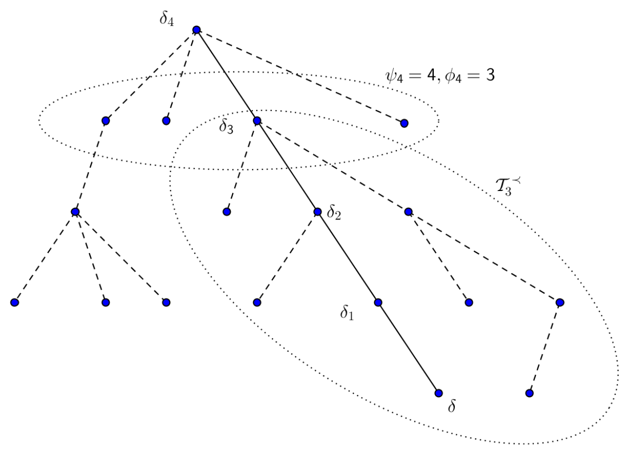

Figure 4.3: A GW-tree conditioned to be of height 4 with the solid line representing the spine and dashed lines representing the subtraps which reach at most level 3 to the left of the spine and at most level 4 to the right of the spine.

vertices) denoting level 0. We then write

$$\Lambda_n := \sum_{j=1}^{\phi_n - 1} \Pi^{n,j,1} + \sum_{j=1}^{\psi_n - \phi_n} \Pi^{n,j,2}$$

to denote the total weight of the subtraps of δ_n then,

$$E^{\mathcal{T}^{\prec}}[\mathcal{R}_\infty] = 2\sum_{n=0}^{\infty} \beta^{-n}(1 + \Lambda_n) \tag{4.31}$$

is the expected time $\mathcal{R}_\infty$ taken for a walk on $\mathcal{T}^{\prec}$ started from δ to return to δ.

Lemma 4.5.1. *Suppose that ξ belongs to the domain of attraction of a stable law of index $\alpha \in (1, 2]$ and $\beta\mu > 1$ then*

$$\mathbb{E}[\mathcal{R}_\infty] < \infty.$$

Proof. Since $\beta > 1$ we have that $2\sum_{n=0}^{\infty} \beta^{-n} = 2/(1 - \beta^{-1})$ thus by (4.31) it suffices to find an appropriate bound on $\mathbf{E}[\Lambda_n]$.

Conditioning the height of the trap to be small reduces the weight therefore $\mathbf{E}[\Pi^{n,j,1}] \leq \mathbf{E}[\Pi^{n,j,2}]$; it then follows from independence of ψ_n and $\Pi^{n,j,2}$ that

$$\mathbf{E}[\Lambda_n] \; = \; \mathbf{E}\left[\sum_{j=1}^{\phi_n - 1} \Pi^{n,j,1} + \sum_{j=1}^{\psi_n - \phi_n} \Pi^{n,j,2}\right] \; \leq \; \mathbf{E}[\Pi^{n,1,2}]\mathbf{E}[\psi_n]. \tag{4.32}$$

Using that conditioning the height of a GW-tree $\mathcal{T}^f$ to be small only decreases the expected generation sizes and that $\mu\beta > 1$, by (2.7)

$$\mathbf{E}[\Pi^{n,1,2}] \;=\; 2\sum_{k=1}^{n}\beta^k\mathbf{E}[Z_k|\mathcal{H}(\mathcal{T}^f)\leq n] \;\leq\; c(\beta\mu)^n \tag{4.33}$$

for some constant c where Z_k are the generation sizes of $\mathcal{T}^f$. Summing over j in (4.30) shows that $\mathbf{P}(\psi_{n+1}=k)=\mathbf{P}(Z_1=k|\mathcal{H}(\mathcal{T}^f)=n+1)$. Recalling that $s_n = \mathbf{P}(\mathcal{H}(\mathcal{T}^f)<n)$, we have that

$$\mathbf{E}[\psi_{n+1}] \;=\; \mathbf{E}[Z_1|\mathcal{H}(\mathcal{T}^f)=n+1] \;=\; \sum_{k=1}^{\infty}kp_k\left(\frac{s_{n+1}^k-s_n^k}{s_{n+2}-s_{n+1}}\right).$$

By (2.9) $1-s_{n+1}\sim c\mu^n$ for some positive constant c. Let $\epsilon>0$ be such that $1-\epsilon-\mu(1+\epsilon)>0$, then for n large we have that $(1-\epsilon)c\mu^n\leq 1-s_{n+1}\leq(1+\epsilon)c\mu^n$. Therefore,

$$s_{n+2}-s_{n+1}=(1-s_{n+1})-(1-s_{n+2})\geq(1-\epsilon-\mu(1+\epsilon))\,c\mu^n\geq C(1-s_n)$$

for some positive constant C. In particular, when $\sigma^2<\infty$, there exists some constant c such that

$$\sum_{k=1}^{\infty}kp_k\left(\frac{s_{n+1}^k-s_n^k}{s_{n+2}-s_{n+1}}\right)\leq c\sum_{k=1}^{\infty}kp_k\left(\frac{1-s_n^k}{1-s_n}\right)\leq c\sigma^2$$

where the final inequality comes from that $(1-s^k)(1-s)^{-1}$ is increasing in s and converges to k for any $k\geq 1$. It therefore follows that $\mathbf{E}[\Lambda_n]\leq C(\beta\mu)^n$ so indeed

$$\mathbb{E}[\mathcal{R}_\infty]\leq C\sum_{n=0}^{\infty}\beta^{-n}(\beta\mu)^n<\infty.$$

When ξ has infinite variance but belongs to the domain of attraction of a stable law

$$\sum_{k=1}^{\infty}kp_k(s_{n+1}^k-s_n^k)=\mu\left(\left(1-\frac{s_nf'(s_n)}{\mu}\right)-\left(1-\frac{s_{n+1}f'(s_{n+1})}{\mu}\right)\right)$$

hence by (4.9) as $n\to\infty$ we have that $\mathbf{E}[\psi_{n+1}]\sim c\mu^{n(\alpha-2)}L_2(\mu^n)$. Combining this with (4.32) and (4.33) we have

$$\mathbf{E}[\Lambda_n] \;\leq\; C(\beta\mu)^n\mu^{n(\alpha-2)}L_2(\mu^n) \;=\; C(\beta\mu^{\alpha-1})^nL_2(\mu^n) \tag{4.34}$$

therefore using (4.31) for C chosen sufficiently large we have that

$$\mathbb{E}[\mathcal{R}_\infty] \leq C\left(1+\sum_{n=1}^{\infty}\mu^{n(\alpha-1)}L_2(\mu^n)\right) < \infty.$$

□

We therefore have that the expected time taken for a walk started from the apex in a trap (of height $\mathscr{H}$) to return to the apex is bounded above by $\mathbb{E}[\mathcal{R}_\infty] < \infty$ independently of its height. Recall that τ_x^+ is the first return time to x and that Lemma 2.3.4 gives the probabilities of reaching the apex in a trap, escaping the trap from the apex and the transition probabilities for the walk in the trap conditional on reaching the apex before escaping. Since the first two probabilities are independent of the structure of the tree except for the height we write

$$p_1(\mathscr{H}) := \frac{1-\beta^{-1}}{1-\beta^{-(\mathscr{H}+1)}}$$

to be the probability that the walk reaches the deepest vertex in the tree before returning to the root starting from the bud and

$$p_2(\mathscr{H}) := \frac{1-\beta^{-1}}{\beta^{\mathscr{H}}-\beta^{-1}} \tag{4.35}$$

to be the probability of escaping from the tree.

For the remainder of the section we will consider only the case that the offspring distribution belongs to the domain of attraction of some stable law of index $\alpha \in (1,2)$. The first aim is to prove Proposition 4.5.2 which shows that the time on excursions in deep traps essentially consists of some geometric number of excursions from the apex to itself. We will then conclude with Corollary 4.5.3 which is an adaptation for FVIE and of which we omit the proof.

Recall that ρ_i^+ is the root of the i^{th} large branch and $\tilde{\chi}_n^i$ is the time spent in this branch by the i^{th} walk $X_n^{(i)}$. This branch has some number N^i buds which are roots of large traps where, by Proposition 4.1.4, N^i converges to a heavy tailed distribution. Let $\rho_{i,j}^+$ be the bud of the j^{th} large trap $\mathcal{T}_{i,j}^+$ in this branch then $W^{i,j} := |\{m \geq 0 : X_{m-1}^{(i)} = \rho_i^+, X_m^{(i)} = \rho_{i,j}^+\}|$ is the number of times that the j^{th} large trap in the i^{th} large branch is visited by the i^{th} copy of the walk. Let $\omega^{(i,j,0)} := 0$ then for $k \leq W^{i,j}$ write $\omega^{(i,j,k)} := \min\{m > \omega^{(i,j,k-1)} : X_{m-1}^{(i)} = \rho_i^+, X_m^{(i)} = \rho_{i,j}^+\}$ to be the start time of the k^{th} excursion into $\mathcal{T}_{i,j}^+$ and $T^{(i,j,k)} := |\{m \in [\omega^{(i,j,k)}, \omega^{(i,j,k+1)}) : X_m^{(i)} \in \mathcal{T}_{i,j}^+\}|$ its duration.

We can then write the time spent in large traps of the i^{th} large branch as

$$\tilde{\chi}_n^i = \sum_{j=1}^{N^i} \sum_{k=1}^{W^{i,j}} T^{(i,j,k)}.$$

For $0 \leq k \leq \mathcal{H}(\mathcal{T}_{i,j}^+)$ write $\delta_k^{(i,j)}$ to be the spinal vertex of distance k from the apex in $\mathcal{T}_{i,j}^+$. Let $T^{*(i,j,k)} := 0$ if there does not exist $m \in [\omega^{(i,j,k)}, \omega^{(i,j,k+1)}]$ such that $X_m = \delta_0^{(i,j)} =: \delta^{(i,j)}$ and

$$\begin{aligned} T^{*(i,j,k)} := & \sup\{m \in [\omega^{(i,j,k)}, \omega^{(i,j,k+1)}] : \ X_m^{(i)} = \delta^{(i,j)}\} \\ & - \inf\{m \in [\omega^{(i,j,k)}, \omega^{(i,j,k+1)}] : \ X_m^{(i)} = \delta^{(i,j)}\} \end{aligned}$$

otherwise to be the duration of the k^{th} excursion into $\mathcal{T}_{i,j}^+$ without the first passage to the apex and the final passage from the apex to the exit. We can then define

$$\tilde{\chi}_n^{i*} := \sum_{j=1}^{N^i} \sum_{k=1}^{W^{i,j}} T^{*(i,j,k)}$$

to be the time spent in the i^{th} large trap without the first passage to and last passage from $\delta^{(i,j)}$ on each excursion (and without the excursions which do not reach $\delta^{(i,j)}$). We want to show that the difference between this and $\tilde{\chi}_n^i$ is negligible. In particular, recalling that $\mathcal{D}_n^{(n)}$ is the collection of large branches by level n, we will show that for all $t > 0$ as $n \to \infty$

$$\mathbb{P}\left(\left| \sum_{i=1}^{|\mathcal{D}_n^{(n)}|} \left(\tilde{\chi}_n^i - \tilde{\chi}_n^{i*} \right) \right| \geq t a_n^{\frac{1}{\gamma}} \right) \to 0.$$

Recall $h_{n,\varepsilon}^+$ from Definition 4.2.3 then, for $\epsilon > 0$, denote by

$$A_6(n) := \bigcap_{i=0}^{n} \{\mathcal{H}(\mathcal{T}_{\rho_i}^{*-}) \leq h_{n,\epsilon}^+\} \tag{4.36}$$

the event that there are no $h_{n,\epsilon}^+$-branches by level n. Using a union bound and (4.12) we have that $\mathbf{P}(A_6(n)^c) \leq n\mathbf{P}(\mathcal{H}(\mathcal{T}_{\rho_0}^{*-}) > h_{n,\epsilon}^+) \to 0$ as $n \to \infty$.

Write

$$A_7(n) := \bigcap_{i=0}^{|\mathcal{D}_n^{(n)}|} \{N^i \leq n^{\frac{2\varepsilon}{\alpha-1}}\} \tag{4.37}$$

to be the event that all large branches up to level n of the backbone have fewer than $n^{\frac{2\varepsilon}{\alpha-1}}$ large traps. Conditional on the number of buds, the number of large traps in the branch follows a binomial distribution conditioned to be at least 1. Since the probability of having 0 successes from $n^{\frac{1+\varepsilon/2}{\alpha-1}}$ trials of probability $\mathbf{P}(\mathcal{H}(\mathcal{T}^{*-}) > h_{n,\varepsilon})$ decays exponentially in n we then have that $\mathbf{P}(N^i \geq n^{\frac{2\varepsilon}{\alpha-1}})$ is bounded above by

$$\frac{\mathbf{P}(\xi^* \geq n^{\frac{1+\varepsilon/2}{\alpha-1}})}{\mathbf{P}(\mathcal{H}(\mathcal{T}^{*-}) > h_{n,\varepsilon})} + \frac{\mathbf{P}\left(Bin\left(n^{\frac{1+\varepsilon/2}{\alpha-1}}, \mathbf{P}(\mathcal{H}(\mathcal{T}^f) \geq h_{n,\varepsilon})\right) \geq n^{\frac{2\varepsilon}{\alpha-1}}\right)}{\mathbf{P}(\mathcal{H}(\mathcal{T}^{*-}) > h_{n,\varepsilon})} + o(n^{-\varepsilon}).$$

By (4.13) we have that $\mathbf{P}(\mathcal{H}(\mathcal{T}^{*-}) \geq h_{n,\varepsilon}) \geq Cn^{-(1-\varepsilon)}$ for n large and some constant C; hence, by (4.11), the first term decays faster than $n^{-\varepsilon}$. Using a Chernoff bound the second term has a stretched exponential decay. Therefore, by Lemma 4.2.4 and a union bound, as $n \to \infty$

$$\mathbf{P}(A_7(n)^c) \leq o(1) + Cn^{\varepsilon}\mathbf{P}(N^i \geq n^{\frac{2\varepsilon}{\alpha-1}}) \to 0.$$

Recall that $d_x := |c(x)|$ is the number of children of x in the tree and define

$$A_8(n) := \bigcap_{i=1}^{|\mathcal{D}_n^{(n)}|} \bigcap_{j=1}^{N^i} \left\{ \sum_{k=0}^{\mathcal{H}(\mathcal{T}_{i,j}^+)} d_{\delta_k^{(i,j)}} \leq n^{3\varepsilon/(\alpha-1)^2} \right\}$$

to be the event that there are fewer than $n^{3\varepsilon/(\alpha-1)^2}$ subtraps on the spine in any large trap. For Z_n, the generation sizes associated to GW-tree $\mathcal{T}^f$, we have that $\mathbf{P}(Z_1 \geq n | \mathcal{H}(\mathcal{T}^f) \geq m)$ is non-decreasing in m; therefore, the number of offspring from a vertex on the spine of a trap can be stochastically dominated by the size biased distribution. Using this and Lemma 4.2.4 with the bounds on A_6 and A_7 we then have that for some slowly varying function $\overline{L}$

$$\begin{aligned}
\mathbf{P}(A_8(n)^c) &\leq o(1) + Cn^{\varepsilon}n^{\frac{2\varepsilon}{\alpha-1}}\mathbf{P}\left(\sum_{k=0}^{h_{n,\epsilon}^+} \xi_k^* \geq n^{3\varepsilon/(\alpha-1)^2}\right) \\
&\leq o(1) + Cn^{\varepsilon}n^{\frac{2\varepsilon}{\alpha-1}} h_{n,\epsilon}^+ \mathbf{P}(\xi^* \geq n^{3\varepsilon/(\alpha-1)^2}/h_{n,\epsilon}^+) \\
&\leq o(1) + n^{\varepsilon}n^{-\frac{\varepsilon}{\alpha-1}}\overline{L}(n)
\end{aligned}$$

where $(\xi_k^*)_{k\geq 1}$ are independent variables with the size biased law; thus $\mathbf{P}(A_8(n)^c) \to 0$ as $n \to \infty$.

Proposition 4.5.2. *In IVIE, for any $t > 0$ as $n \to \infty$*

$$\mathbb{P}\left(\left|\sum_{i=1}^{|\mathcal{D}_n^{(n)}|} \left(\tilde{\chi}_n^i - \tilde{\chi}_n^{i*}\right)\right| \geq ta_n^{\frac{1}{\gamma}}\right) \to 0.$$

Proof. Let $A'(n) := \bigcap_{i=1}^8 A_i(n)$ (where $A_1(n) := A_1(n,T)$, $A_2(n) := \tilde{A}_2(n,T)$ and

$A_4(n) := A_4(n,T)$) then using the bounds on A_i for $i = 1,...,8$ it follows that $\mathbb{P}(A'(n)^c) \to 0$ as $n \to \infty$. In particular, on $A_1(n)$ (from (4.14)) we have that $|\mathcal{D}_n^{(n)}| \leq Cn^{\varepsilon}$ and on $A_7(n)$ (from (4.37)) we have that $N^i \leq n^{\frac{2\varepsilon}{\alpha-1}}$ for all i therefore by Markov's inequality

$$\begin{aligned}
&\mathbb{P}\left(\left|\sum_{i=1}^{|\mathcal{D}_n^{(n)}|}\left(\tilde{\chi}_n^i - \tilde{\chi}_n^{i*}\right)\right| \geq ta_n^{\frac{1}{\gamma}}\right) \\
&\qquad\qquad \leq \mathbb{P}(A'(n)^c) + \frac{1}{ta_n^{\frac{1}{\gamma}}}\mathbb{E}\left[\mathbf{1}_{A'(n)}\sum_{i=1}^{|\mathcal{D}_n^{(n)}|}\sum_{j=1}^{N^i}\sum_{k=1}^{W^{i,j}}\left(T_n^{(i,j,k)} - T_n^{*(i,j,k)}\right)\right] \\
&\qquad\qquad \leq o(1) + \frac{Cn^{\varepsilon\left(\frac{\alpha+1}{\alpha-1}\right)}}{ta_n^{\frac{1}{\gamma}}}\mathbb{E}\left[\mathbf{1}_{A'(n)}\sum_{k=1}^{W^{(1,1)}}\left(T_n^{(1,1,k)} - T_n^{*(1,1,k)}\right)\right] \qquad (4.38)
\end{aligned}$$

where we recall that $T_n^{*(i,j,k)} \leq T_n^{(i,j,k)}$ for all i,j,k.

Since, by Lemma 2.3.3, the number of excursions $W^{i,j}$ are independent of the excursion times and have marginal distributions of geometric random variables with parameter $(\beta-1)/(2\beta-1)$ we have that

$$\mathbb{E}\left[\mathbf{1}_{A'(n)}\sum_{k=1}^{W^{(1,1)}}\left(T_n^{(1,1,k)} - T_n^{*(1,1,k)}\right)\right] = \mathbb{E}[W^{(1,1)}]\mathbb{E}\left[\mathbf{1}_{A'(n)}\left(T_n^{(1,1,1)} - T_n^{*(1,1,1)}\right)\right].$$

For a given excursion either the walk reaches the apex $\delta^{(1,1)}$ before returning to the root $\rho_{1,1}^+$ or it does not. In the first case the difference $T_n^{(1,1,1)} - T_n^{*(1,1,1)}$ is the time taken to reach $\delta^{(1,1)}$ conditional on the walker reaching $\delta^{(1,1)}$ before $\rho_{1,1}^+$ added to the time taken to reach $\rho_{1,1}^+$ from $\delta^{(1,1)}$ conditional on reaching $\rho_{1,1}^+$ before returning to $\delta^{(1,1)}$. In the second case the difference is the time taken to return to the root given that the walker returns to the root without reaching $\delta^{(1,1)}$. In particular, recalling that $\mathcal{T}_{1,1}^+$ is the trap rooted at $\rho_{1,1}^+$ we have that

$$\begin{aligned}
\mathbb{E}[\mathbf{1}_{A'(n)}(T_n^{(1,1,1)} - T_n^{*(1,1,1)})] \leq\; &\mathbf{E}\left[\mathbf{1}_{A'(n)}E_{\rho_{1,1}^+}^{\mathcal{T}_{1,1}^+}[\mathbf{1}_{A'(n)}\tau^+_{\delta(1,1)}|\tau^+_{\delta(1,1)} < \tau^+_{\rho_{1,1}^+}]\right] \qquad (4.39)\\
&+ \mathbf{E}\left[\mathbf{1}_{A'(n)}E_{\delta(1,1)}^{\mathcal{T}_{1,1}^+}[\mathbf{1}_{A'(n)}\tau^+_{\rho_{1,1}^+}|\tau^+_{\rho_{1,1}^+} < \tau^+_{\delta(1,1)}]\right] \\
&+ \mathbf{E}\left[\mathbf{1}_{A'(n)}E_{\rho_{1,1}^+}^{\mathcal{T}_{1,1}^+}[\mathbf{1}_{A'(n)}\tau^+_{\rho_{1,1}^+}|\tau^+_{\rho_{1,1}^+} < \tau^+_{\delta(1,1)}]\right].
\end{aligned}$$

We want to show that each of the terms in (4.39) can be bounded appropriately. This follows similarly to [10, Lemmas 8.2 & 8.3] so we only sketch the details. Conditional on the event that the walk returns to the root of the trap before reaching the apex we have that:

1. the transition probabilities of the walk in subtraps are unchanged;
2. from any vertex on the spine, the walk is more likely to move towards the root than to any vertex in the subtrap;
3. from any vertex on the spine, excluding the root and apex, the probability of moving towards the root is at least β times that of moving towards the apex.

Property 3 above shows that the probability of escaping the trap from any vertex on the spine is at least the probability, p_∞, of a regeneration for the β-biased random walk on $\mathbb{Z}$. From this we have that the number of visits to any spinal vertex can be stochastically dominated by a geometric random variable with parameter p_∞. Similarly, using property 2 above, we see that the number of visits to any subtrap can be stochastically dominated by a geometric random variable with parameter $p_\infty/2$.

Using a union bound with A_1, A_7, A_8 and (2.9) we have that with high probability there are no subtraps of height greater than $h_{n,\varepsilon}$. In particular, by (2.14), the expected time in any subtrap can be bounded above by $C(\beta\mu)^{h_{n,\varepsilon}}$ for some constant C using property 1. From this it follows that

$$\begin{aligned}\mathbf{E}\left[\mathbf{1}_{A'(n)}E_{\rho_{1,1}^+}^{\mathcal{T}_{1,1}^+}[\mathbf{1}_{A'(n)}\tau_{\rho_{1,1}^+}^+|\tau_{\rho_{1,1}^+}^+<\tau_{\delta(1,1)}^+]\right] &\leq \mathbf{E}\left[\mathbf{1}_{A'(n)}E_{\delta(1,1)}^{\mathcal{T}_{1,1}^+}[\mathbf{1}_{A'(n)}\tau_{\rho_{1,1}^+}^+|\tau_{\rho_{1,1}^+}^+<\tau_{\delta(1,1)}^+]\right] \\ &\leq o(1)+h_{n,\varepsilon}^+E[Geo(p_\infty)]+Cn^{\frac{3\varepsilon}{(\alpha-1)^2}}(\beta\mu)^{h_{n,\varepsilon}} \\ &\leq o(1)+C\overline{L}(n)n^{\frac{(1-\varepsilon)}{\alpha-1}\frac{\log(\beta\mu)}{\log(\mu^{-1})}+\frac{3\varepsilon}{(\alpha-1)^2}}\end{aligned}$$

for some constant C and slowly varying function $\overline{L}$.

A symmetric argument shows that the same bound can be achieved for the first term in (4.39). It then follows that the final term in (4.38) can be bounded above by $C_t\hat{L}(n)n^{-\frac{1}{\alpha-1}+\tilde{\varepsilon}}$ where $\tilde{\varepsilon}$ can be made arbitrarily small by choosing ε sufficiently small. □

A straightforward adaptation of [10, Proposition 8.1] (similar to the previous calculation) shows Corollary 4.5.3 which is the corresponding result for FVIE.

Corollary 4.5.3. *In FVIE, for any $t>0$ as $n\to\infty$*

$$\mathbb{P}\left(\left|\sum_{i=1}^{|\mathcal{D}_n^{(n)}|}\left(\tilde{\chi}_n^i-\tilde{\chi}_n^{i*}\right)\right|\geq tn^{\frac{1}{\gamma}}\right)\to 0.$$

By Proposition 4.5.2 and Corollary 4.5.3, in FVIE and IVIE, almost all time up to the walk reaching level n is spent on excursions from the apex in deep traps. The aim of the remainder of the section is to prove Proposition 4.5.14 which shows that the time spent on the excursions from the apex in a single large branch (suitably scaled) converges in distribution along the given subsequences. To ease notation, for the remainder of the section we work on a pruned dummy branch $\mathcal{T}^*$ so that the time $\tilde{\chi}_n^{i*}$ has the distribution of a sum of excursion times from the apexes of $\mathcal{T}^*$.

Recall from Definition 4.1.1 that $\mathcal{T}^{*-}$ is a dummy branch with root ρ, buds $\rho_1, ..., \rho_{\xi^*-1}$ each of which is the root of an f-GW-tree $\mathcal{T}_j^f$ with height $H_j := \mathcal{H}(\mathcal{T}_j^f)$. We now define a pruned version of this branch which only contains traps of height at least $h_{n,\varepsilon}$.

Definition 4.5.4. *(Pruned dummy branch) Let*

$$N := \sum_{j=1}^{\xi^*-1} \mathbf{1}_{\{H_j \geq h_{n,\varepsilon}\}}$$

denote the number of traps in $\mathcal{T}^{*-}$ of at least critical height. Denote $(\mathcal{T}_j^+)_{j=1}^N$ to be those large traps, $(\rho_j^+)_{j=1}^N$ their roots and $H_j^+ := \mathcal{H}(\mathcal{T}_j^+)$ the height of the j^{th} large trap in the branch. Similarly, let $(\mathcal{T}_j^-)_{j=1}^{\xi^*-1-N}$ denote the small traps, $(\rho_j^-)_{j=1}^{\xi^*-1-N}$ their roots and $H_j^- := \mathcal{H}(\mathcal{T}_j^-)$ the height of the j^{th} small trap in the branch.

Let $\mathcal{T}^*$ be $\mathcal{T}^{*-}$ pruned to consist precisely of the root ρ, buds $(\rho_j^+)_{j=1}^N$ and traps $(\mathcal{T}_j^+)_{j=1}^N$. We write $\overline{H} := \mathcal{H}(\mathcal{T}^*) - 1$ to be the height of the largest trap and for $K \in \mathbb{Z}$ let $\overline{H}_n^K := h_{n,0} + K$ then denote

$$\mathbb{P}^K(\cdot) := \mathbb{P}(\cdot | \overline{H} = \overline{H}_n^K) \qquad \text{and} \qquad \mathbf{P}^K(\cdot) := \mathbf{P}(\cdot | \overline{H} = \overline{H}_n^K).$$

Write W^j to be the total number of excursions into $\mathcal{T}_j^+$ and B^j the number of excursions which reach the apex δ^j.

For each $k \leq B^j$ we define $G^{j,k}$ to be the number of return times to δ^j on the k^{th} excursion which reaches δ^j.

For $l = 1, ..., G^{j,k}$ let $\mathcal{R}^{j,k,l}$ denote the duration of the l^{th} excursion from δ^j to itself on the k^{th} excursion into $\mathcal{T}_j^+$ which reaches δ^j.

The height of the branch and the total number of traps in the branch have a strong relationship. Lemma 4.5.5 shows the exact form of this relationship in the limit as $n \to \infty$. Recall from (2.9) that c_μ is the positive constant such that $\mathbf{P}(\mathcal{H}(\mathcal{T}^f) \geq n) \sim c_\mu \mu^n$ as $n \to \infty$ then write

$$b_n^K := \frac{\mu^{-\overline{H}_n^K}}{c_\mu}. \tag{4.40}$$

Lemma 4.5.5. *In IVIE, under $\mathbf{P}^K$ we have that the sequence of random variables $(\xi^* - 1)/b_n^K$ converge in distribution to a random variable $\overline{\xi}$ satisfying*

$$\mathbf{P}(\overline{\xi} \geq t) = \frac{\alpha - 1}{\Gamma(2-\alpha)(1-\mu^{\alpha-1})} \int_t^\infty y^{-\alpha}(e^{-\mu y} - e^{-y}) \mathrm{d}y.$$

Proof. We prove this by showing the convergence of

$$\mathbf{P}\left(\xi^* - 1 \geq tb_n^K | \overline{H} = \overline{H}_n^K\right) = \mathbf{P}\left(\overline{H} = \overline{H}_n^K | \xi^* - 1 \geq tb_n^K\right) \frac{\mathbf{P}(\xi^* - 1 \geq tb_n^K)}{\mathbf{P}(\overline{H} = \overline{H}_n^K)} \tag{4.41}$$

for all $t > 0$. To begin we consider $\mathbf{P}\left(\overline{H} = \overline{H}_n^K | \xi^* - 1 \geq tb_n^K\right)$.

The heights of individual traps are independent under this conditioning hence

$$\mathbf{P}\left(\overline{H} \leq \overline{H}_n^K | \xi^* - 1 \geq tb_n^K\right) = \mathbf{E}\left[\mathbf{P}(\mathcal{H}(\mathcal{T}^f) \leq \overline{H}_n^K)^{\xi^*-1} | \xi^* - 1 \geq tb_n^K\right].$$

We know the asymptotic form of $\mathbf{P}(\mathcal{H}(\mathcal{T}^f) \leq \overline{H}_n^K)$ from (2.9) thus we need to consider the distribution of $\xi^* - 1$ conditioned on $\xi^* - 1 \geq tb_n^K$. By the tail formula for $\xi^* - 1$, following Definition 4.0.3, we have that for $r \geq 1$ as $n \to \infty$

$$\mathbf{P}\left(\frac{\xi^* - 1}{tb_n^K} \geq r \middle| \xi^* - 1 \geq tb_n^K\right) = \frac{\mathbf{P}(\xi^* - 1 \geq rtb_n^K)}{\mathbf{P}(\xi^* - 1 \geq tb_n^K)} \sim r^{-(\alpha-1)}.$$

We therefore have that, conditional on $\xi^* - 1 \geq tb_n^K$, the sequence $(\xi^* - 1)/tb_n^K$ converges in distribution to a variable Y with tail $\mathbf{P}(Y \geq r) = r^{-(\alpha-1)} \wedge 1$. Using the form of b_n^K we then have that

$$\mathbf{P}(\mathcal{H}(\mathcal{T}^f) \leq \overline{H}_n^K)^{tb_n^K} = e^{-t\mu(1+o(1))}.$$

It therefore follows that

$$\lim_{n\to\infty} \mathbf{P}\left(\overline{H} \leq \overline{H}_n^K | \xi^* - 1 \geq tb_n^K\right) = \mathbf{E}[e^{-t\mu Y}].$$

Repeating with $\overline{H}_n^K$ replaced by $\overline{H}_n^K - 1$ we have that $\mathbf{P}(\overline{H} = \overline{H}_n^K | \xi^* - 1 \geq tb_n^K) \to \mathbf{E}[e^{-t\mu Y}] - \mathbf{E}[e^{-tY}]$ as $n \to \infty$. For $\theta > 0$

$$\mathbf{E}[e^{-\theta tY}] = (\alpha - 1)t^{\alpha-1} \int_t^\infty e^{-\theta y} y^{-\alpha} \mathrm{d}y$$

therefore

$$\lim_{n\to\infty} \mathbf{P}\left(\overline{H} = \overline{H}_n^K | \xi^* - 1 \geq tb_n^K\right) = (\alpha - 1)t^{\alpha-1} \int_t^\infty y^{-\alpha}(e^{-\mu y} - e^{-y})\mathrm{d}y. \tag{4.42}$$

By (4.12) we have that as $n \to \infty$

$$\begin{aligned}
\mathbf{P}\left(\overline{H} = \overline{H}_n^K\right) &= \mathbf{P}\left(\mathcal{H}\left(\mathcal{T}^{*-}\right) > \overline{H}_n^K\right) - \mathbf{P}\left(\mathcal{H}\left(\mathcal{T}^{*-}\right) > \overline{H}_n^K + 1\right) \\
&\sim \Gamma\left(2 - \alpha\right) c_\mu^{\alpha-1} \left(1 - \mu^{\alpha-1}\right) \mathbf{P}\left(\xi^* - 1 \geq \mu^{-\overline{H}_n^K}\right) \\
&= \Gamma\left(2 - \alpha\right) c_\mu^{\alpha-1} \left(1 - \mu^{\alpha-1}\right) \mathbf{P}\left(\xi^* - 1 \geq c_\mu b_n^K\right)
\end{aligned}$$

therefore

$$\begin{aligned}\frac{\mathbf{P}(\xi^*-1\geq tb_n^K)}{\mathbf{P}(\overline{H}=\overline{H}_n^K)} &\sim \frac{\mathbf{P}(\xi^*-1\geq tb_n^K)}{\Gamma(2-\alpha)(1-\mu^{\alpha-1})c_\mu^{\alpha-1}\mathbf{P}(\xi^*-1\geq c_\mu b_n^K)} \\ &\sim \frac{t^{-(\alpha-1)}}{\Gamma(2-\alpha)(1-\mu^{\alpha-1})}.\end{aligned}$$

Combining this with (4.42) in (4.41) we have that

$$\lim_{n\to\infty}\mathbf{P}(\xi^*-1\geq tb_n^K|\overline{H}=\overline{H}_n^K)=\frac{\alpha-1}{\Gamma(2-\alpha)(1-\mu^{\alpha-1})}\int_t^\infty y^{-\alpha}(e^{-\mu y}-e^{-y})\mathrm{d}y.$$

□

Notice that under $\mathbf{P}$ the pruned dummy branch $\mathcal{T}^*$ is the single vertex ρ with high probability however under $\mathbf{P}^K$ there is at least one trap. By Lemma 2.3.3, conditional on N, $(W^j)_{j=1}^N$ have a joint negative multinomial distribution. Moreover, W^j and B^j are coupled so that B^j is binomially distributed with W^j trials and success probability $p_1(H_j^+)$. The number $G^{j,k}$ of return times to δ^j is geometrically distributed with failure probability $p_2(H_j^+)$. It follows that each $\tilde{\chi}_n^{i*}$ is equal in distribution to

$$\chi_n^*:=\sum_{j=1}^N\sum_{k=1}^{B^j}\sum_{l=1}^{G^{j,k}}\mathcal{R}^{j,k,l}.$$

Define the scaled excursion time in large traps of a large branch as

$$\zeta^{(n)}:=\chi_n^*\beta^{-\overline{H}}=\beta^{-\overline{H}}\sum_{j=1}^N\sum_{k=1}^{B^j}\sum_{l=1}^{G^{j,k}}\mathcal{R}^{j,k,l} \tag{4.43}$$

then we will show that $\zeta^{(n)}$ converges in distribution under $\mathbb{P}^K$ along subsequences $n_l(t)$. Lemma 4.5.6 gives an upper bound on the number of large traps in a branch conditioned on its height.

Lemma 4.5.6. *For any $\epsilon>0$ and $K\in\mathbb{Z}$*

$$\lim_{n\to\infty}\mathbf{P}^K\left(N\geq n^{\frac{\varepsilon+\epsilon}{\alpha-1}}\right)=0.$$

Proof. Conditioned on the height of the branch and number of buds we have that at least one trap attains the maximum height, all others have the distribution of heights of GW-tree conditioned on their maximum height therefore

$$\mathbf{P}^K\left(N\geq n^{\frac{\varepsilon+\epsilon}{\alpha-1}}\right)\leq\mathbf{P}^K(\xi^*-1\geq\log(n)b_n^K)+\mathbf{P}\left(N\geq n^{\frac{\varepsilon+\epsilon}{\alpha-1}}-1|\xi^*-1=\log(n)b_n^K\right). \tag{4.44}$$

By Lemma 4.5.5 $\mathbf{P}^K\left(\xi^* - 1 \geq \log(n)b_n^K\right)$ converges to 0 as $n \to \infty$. Conditioned on having $\xi^* - 1 = \log(n)b_n^K$ buds we have that N is binomially distributed with $\log(n)b_n^K$ trails and success probability $\mathbf{P}(\mathcal{H}(\mathcal{T}^f) \geq h_{n,\varepsilon}) \leq C\mu^{h_{n,\varepsilon}}$ by (2.9). Since for some slowly varying function $\overline{L}$ we have that

$$\mathbf{E}\left[Bin\left(\log(n)b_n^K,\ C\mu^{h_{n,\varepsilon}}\right)\right] \leq C\mu^K \log(n)\frac{a_n}{a_{n^{1-\varepsilon}}} \leq \overline{L}(n)\mu^K n^{\frac{\varepsilon}{\alpha-1}},$$

a Chernoff bound shows that the final term in (4.44) converges to 0. $\square$

For $\tilde{\varepsilon} > 0$ write

$$A_9(n) = \bigcap_{j=1}^{N}\left\{1 \leq \frac{\beta^{H_j^+}}{1-\beta^{-1}}E[G^{j,1}]^{-1} \leq 1 + \tilde{\varepsilon}\right\}.$$

Recall from (4.35) that $p_2(\mathscr{H})$ is the probability that a walk started from the apex of a tree of height $\mathscr{H}$ reaches the root before returning to the apex. Since $G^{j,k}$ are independent geometric random variables there exist independent exponential random variables $e_{j,k}$ such that

$$G^{j,k} = \left\lfloor \frac{e_{j,k}}{-\log(1-p_2(H_j^+))}\right\rfloor \sim Geo(p_2(H_j^+)).$$

By (4.35) we then have that

$$E[G^{j,1}] = \left(1 - \frac{1-\beta^{-1}}{\beta^{H_j^+} - \beta^{-1}}\right)\left(1 - \beta^{-(H_j^+ + 1)}\right)\frac{\beta^{H_j^+}}{1-\beta^{-1}} \tag{4.45}$$

therefore, since $H_j^+ \geq h_{n,\varepsilon}$, for any $\tilde{\varepsilon} > 0$ there exists n large such that $\mathbf{P}^K(A_9(n)) = 1$ for any $K \in \mathbb{Z}$.

Recall from (4.35) and Definition 4.5.4 that $G^{j,k}$ is geometrically distributed with failure probability $p_2(H_j^+) \geq p_2(h_{n,\varepsilon})$. Write

$$A_{10}^{(j,k)}(n) := \left\{(1-\tilde{\varepsilon})G^{j,k} \leq E[G^{j,k}]e_{j,k} \leq (1+\tilde{\varepsilon})G^{j,k}\right\}.$$

For p sufficiently small we have that

$$\left|\frac{1-p}{p} - \frac{1}{-\log(1-p)}\right|$$

is bounded below by some positive constant $M > 0$ therefore $P(A_{10}^{(j,k)}(n)^c)$ is bounded above by

$$\begin{aligned}
&P\left(\left(\frac{1-p_2(H_j^+)}{p_2(H_j^+)}-\frac{1-\tilde{\varepsilon}}{-\log(1-p_2(H_j^+))}\right)e_{j,k}<1\right)\\
&\quad+P\left(\left(\frac{1-p_2(H_j^+)}{p_2(H_j^+)}-\frac{1+\tilde{\varepsilon}}{-\log(1-p_2(H_j^+))}\right)e_{j,k}>-1\right)\\
&\quad\leq P\left(e_{j,k}\left(C_{\tilde{\varepsilon}}p_2(H_j^+)^{-1}-M\right)<1\right)+P\left(e_{j,k}\left(M-C_{\tilde{\varepsilon}}p_2(H_j^+)^{-1}\right)>-1\right)\\
&\quad\leq 2P\left(e_{j,k}<C_{M,\tilde{\varepsilon}}p_2(H_j^+)\right).
\end{aligned}$$

In particular, we have that there exists a constant $\tilde{C}$ such that for any $\tilde{\varepsilon}>0$ there exists n large such that

$$P(A_{10}^{(j,k)}(n)^c)\leq \tilde{C}p_2(h_{n,\varepsilon})\leq \tilde{C}a_{n^{1-\varepsilon}}^{-1/\gamma}.$$

By Definition 4.5.4 we have that $B^j\leq W^j$. Moreover $N\leq n^{\frac{\varepsilon+\tilde{\varepsilon}}{\alpha-1}}$ with high probability for any $\tilde{\epsilon}>0$ by Lemma 4.5.6 and $W^j\leq C\log(n)$ for all j by the bound on the event $A_5(n)^c$ (from (4.16)). Therefore, writing

$$A_{10}(n):=\bigcap_{j=1}^{N}\bigcap_{k=1}^{B^j}A_{10}^{(j,k)}(n)$$

a union bound gives us that $P(A_{10}(n)^c)\to 0$ as $n\to\infty$.

By comparison with the biased random walk on $\mathbb{Z}$ we have that $p_1(H_j^+)\geq p_\infty=1-\beta^{-1}$ therefore we can define a random variable $B_\infty^j\sim Bin(B^j,p_\infty/p_1(H_j^+))$. It then follows that $B^j\geq B_\infty^j\sim Bin(W^j,p_\infty)$ and

$$p_1(H_j^+)-p_\infty=\frac{1-\beta^{-1}}{1-\beta^{-(H_j^++1)}}-(1-\beta^{-1})\leq\beta^{-H_j^+}. \tag{4.46}$$

Write

$$A_{11}(n):=\bigcap_{j=1}^{N}\left\{B^j=B_\infty^j\right\}.$$

Since the marginal distribution of W^1 does not depend on n, using (4.46), the bound on N from Lemma 4.5.6 and the coupling between B^1 and B_∞^1 we have that

$$\begin{aligned}
\mathbb{P}^K(A_{11}(n)^c)&\leq o(1)+n^{\frac{\varepsilon+\tilde{\varepsilon}}{\alpha-1}}\sum_{k=0}^{\infty}\mathbb{P}(W^1=k)\mathbb{P}(B^1\neq B_\infty^1|W^1=k)\\
&\leq o(1)+n^{\frac{\varepsilon+\tilde{\varepsilon}}{\alpha-1}}\sum_{k=0}^{\infty}\mathbb{P}(W^1=k)k(p_1(H_1^+)-p_\infty)\\
&\leq o(1)+n^{\frac{\varepsilon+\tilde{\varepsilon}}{\alpha-1}}\beta^{-h_{n,\varepsilon}}\mathbb{E}[W^1] \qquad (4.47)
\end{aligned}$$

which decays to 0 as $n \to \infty$.

By choosing $\varepsilon > 0$ sufficiently small we can choose κ in the range $\varepsilon(1/\gamma + 1/(\alpha-1)) < \kappa < \min\{2(\alpha-1),\ 1/\gamma\}$ then write

$$A_{12}(n) := \bigcap_{j=1}^{N} \{E[(\mathcal{R}_n^{j,1,1})^2] < n^{\frac{\gamma^{-1}-\kappa}{\alpha-1}}\}$$

to be the event that there are no large traps with expected squared excursion time too large.

Lemma 4.5.7. *In IVIE, for any $K \in \mathbb{Z}$, as $n \to \infty$ we have that $\mathbb{P}^K(A_{12}(n)^c) \to 0$.*

Proof. Recall from (4.36) that, for $\epsilon > 0$, $A_6(n)$ is the event that all large branches are shorter than $h_{n,\epsilon}^+$ and since $N \leq n^{\frac{\varepsilon+\tilde{\varepsilon}}{\alpha-1}}$ with high probability we have that

$$\mathbb{P}(A_{12}(n)^c) \leq o(1) + n^{\frac{\varepsilon+\tilde{\varepsilon}}{\alpha-1}} \mathbb{P}\left(\mathbf{1}_{\{A_6(n)\}} E[(\mathcal{R}_n^{1,1,1})^2]^{1/2} > n^{\frac{\gamma^{-1}-\kappa}{2(\alpha-1)}}\right).$$

The method used to prove [10, Lemma 9.1] holds for any fixed tree; in particular, it yields the following upper bound:

$$E[(\mathcal{R}_n^{1,1,1})^2]^{1/2} \leq C \sum_{y \in \mathcal{T}_1^+} \beta^{d(y,\delta_1^+)/2} \pi(y)$$

where π is the invariant measure scaled so that $\pi(\delta_1^+) = 1$ and d denotes the graph distance.

We then have that

$$\begin{aligned}
\mathbf{E}\left[\mathbf{1}_{\{A_6(n)\}} E[(\mathcal{R}_n^{(1,1,1)})^2]^{1/2}\right] &\leq C\mathbf{E}\left[\mathbf{1}_{\{A_6(n)\}} \sum_{y \in \mathcal{T}_1^+} \beta^{d(y,\delta_1^+)/2} \pi(y)\right] \\
&\leq C\mathbf{E}\left[\mathbf{1}_{\{A_6(n)\}} \sum_{i \geq 1} \beta^{i/2} \beta^{-i}(1+\Lambda_i)\right] \\
&\leq C \sum_{i=0}^{h_{n,\epsilon}^+} (\beta^{1/2} \mu^{\alpha-1-\epsilon})^i
\end{aligned}$$

where the final inequality follows by (4.34). If $\beta^{1/2}\mu^{\alpha-1-\epsilon} \leq 1$ then by Markov's inequality we have that $\mathbb{P}^K(A_{12}(n)^c) \to 0$ as $n \to \infty$ since $\kappa < \gamma^{-1}$. Otherwise by Markov's inequality

$$\begin{aligned}
\mathbb{P}^K(A_{12}(n)^c) &\leq o(1) + Cn^{\frac{\varepsilon+\tilde{\varepsilon}}{\alpha-1}} (\beta^{1/2}\mu^{\alpha-1-\epsilon})^{h_{n,\epsilon}^+} n^{\frac{\kappa-\gamma^{-1}}{2(\alpha-1)}} \\
&\leq o(1) + \overline{L}(n) n^{\frac{\kappa}{2(\alpha-1)} - 1 + \frac{\epsilon}{\alpha-1}\left(\frac{1}{2\gamma}+2-\alpha+\epsilon\right) + \frac{\varepsilon+\tilde{\varepsilon}}{\alpha-1}}
\end{aligned}$$

for some slowly varying function $\overline{L}$. In particular, since $\kappa < 2(\alpha-1)$ we can choose $\epsilon, \varepsilon, \tilde{\epsilon}$ sufficiently small such that this converges to 0 as $n \to \infty$. $\square$

Write

$$A_{13}(n) = \bigcap_{j=1}^{N} \bigcap_{k=1}^{B^j} \left\{ (1-\tilde{\varepsilon})G^{j,k} E[\mathcal{R}_n^{j,1,1}] \leq \sum_{l=1}^{G^{j,k}} \mathcal{R}^{j,k,l} \leq (1+\tilde{\varepsilon})G^{j,k} E[\mathcal{R}_n^{j,1,1}] \right\}$$

to be the event that on each excursion that reaches the apex of a large trap, the total excursion time before leaving the trap is approximately the product of the number of excursions and the expected excursion time.

Lemma 4.5.8. *In IVIE, for any $K \in \mathbb{Z}$, as $n \to \infty$ we have that $\mathbb{P}^K(A_{13}(n)^c) \to 0$.*

Proof. With high probability we have that no trap is visited more than $C\log(n)$ times by (4.16) and also $N \leq n^{\frac{\varepsilon+\tilde{\epsilon}}{\alpha-1}}$ by Lemma 4.5.6. Any excursion is of length at least 2 hence $E[\mathcal{R}_n^{1,1,1}] \geq 2$. Therefore, by Lemma 4.5.7 and Chebyshev's inequality, $\mathbb{P}^K(A_{13}(n)^c)$ is bounded above by a small error added to

$$\begin{aligned} C\log(n)n^{\frac{\varepsilon+\tilde{\epsilon}}{\alpha-1}}\mathbb{P}&\left(\left|\sum_{l=1}^{G^{1,1}} \frac{\mathcal{R}_n^{1,1,l}}{E[\mathcal{R}_n^{1,1,1}]G^{1,1}} - 1\right| > \tilde{\varepsilon}, G^{1,1} > 0, E[(\mathcal{R}_n^{1,1,1})^2] < n^{\frac{\gamma^{-1}-\kappa}{\alpha-1}}\right) \\ &\leq \frac{C\log(n)n^{\frac{\gamma^{-1}+\varepsilon+\tilde{\epsilon}-\kappa}{\alpha-1}}}{\tilde{\varepsilon}^2} E\left[\frac{\mathbf{1}_{\{G^{1,1}>0\}}}{G^{1,1}}\right]. \end{aligned}$$

It then follows that, since $G^{1,1} \sim Geo(p_2(H_1^+))$ (where, from (4.35), $p_2(H)$ is the probability that a walk reaches the apex in the trap of height H) and $p_2(H_1^+) \leq c\beta^{-h_{n,\varepsilon}} = ca_{n^{1-\varepsilon}}^{-\frac{1}{\gamma}}$, we have

$$E\left[\frac{\mathbf{1}_{\{G^{(1,1,1)}>0\}}}{G^{(1,1,1)}}\right] \leq E\left[-\frac{p_2(H_1^+)}{1-p_2(H_1^+)}\log\left(p_2(H_1^+)\right)\right] \leq \overline{L}(n)n^{-\frac{1-\varepsilon}{\gamma(\alpha-1)}}$$

for some slowly varying function $\overline{L}$. In particular,

$$\mathbb{P}^K(A_{13}(n)^c) \leq o(1) + L_{\tilde{\varepsilon}}(n)n^{\frac{\varepsilon\left(\frac{1}{\gamma}+\frac{1}{\alpha-1}\right)+\tilde{\epsilon}-\kappa}{\alpha-1}}$$

which converges to zero by the choice of $\kappa > \varepsilon(1/\gamma + 1/(\alpha-1))$. $\square$

Lemma 4.5.9 demonstrates that the expected time spent on an excursion from the apex of a trap of height at least $h_{n,\varepsilon}$ does not differ too greatly from the expected excursion time in an infinite version of the trap. Let $\mathcal{R}_\infty^j$ be an excursion time from δ_j^+ to itself in an extension of $\mathcal{T}_j^+$ to an infinite trap constructed according to the algorithm at the beginning of the section where $\mathcal{T}^{\prec}_{H_j^+}$ is replaced by $\mathcal{T}_j^+$. Write

$$A_{14}(n) := \bigcap_{j=1}^{N} \left\{ E[\mathcal{R}^j_\infty] - E[\mathcal{R}^{j,k,l}] < \tilde{\varepsilon} \right\}.$$

Lemma 4.5.9. *In IVIE, for any $K \in \mathbb{Z}$ as $n \to \infty$ we have that $\mathbf{P}^K(A_{14}(n)^c) \to 0$.*

Proof. By comparing the transition probabilities of the conditioned and unconditioned walks up to distance $h_{n,\varepsilon}/2$ and crudely discarding the sections of the excursions in the conditioned case which reach further we have that for a constant c and n sufficiently large

$$0 \leq E[\mathcal{R}^j_\infty] - E[\mathcal{R}^{j,k,l}] \leq c\beta^{-h_{n,\varepsilon}/2} \sum_{k=0}^{h_{n,\varepsilon}/2} \beta^{-k}(1+\Lambda_k) + 2 \sum_{k=h_{n,\varepsilon}/2+1}^{\infty} \beta^{-k}(1+\Lambda_k)$$

for all $j = 1, ..., N$ where Λ_k are the weights of the extension of $\mathcal{T}_j^+$. Recall that $N \leq n^{\frac{\varepsilon+\tilde{\epsilon}}{\alpha-1}}$ with high probability by Lemma 4.5.6. By (4.34) we have that $\mathbf{E}[\Lambda_k] \leq C(\beta\mu^{\alpha-1})^k$ therefore using Markov's inequality we have that

$$\begin{aligned}
\mathbf{P}(A_{14}(n)^c) &\leq \frac{Cn^{\frac{\varepsilon+\tilde{\epsilon}}{\alpha-1}}}{\tilde{\varepsilon}} \mathbf{E}[E[\mathcal{R}^j_\infty] - E[\mathcal{R}^{j,1,1}_n]] \\
&\leq C_{\tilde{\varepsilon}} n^{\frac{\varepsilon+\tilde{\epsilon}}{\alpha-1}} \left(\beta^{-\frac{h_{n,\varepsilon}}{2}} \sum_{k=0}^{\infty} \left(\beta^{-k} + \mu^{k(\alpha-1-\tilde{\varepsilon})} \right) + \sum_{k=h_{n,\varepsilon}/2+1}^{\infty} \mu^{k(\alpha-1-\tilde{\varepsilon})} \right) \\
&\leq C_{\tilde{\varepsilon}} n^{\frac{\varepsilon+\tilde{\epsilon}}{\alpha-1}} \left(\beta^{-\frac{h_{n,\varepsilon}}{2}} + \mu^{h_{n,\varepsilon}\frac{(\alpha-1-\tilde{\varepsilon})}{2}} \right).
\end{aligned}$$

Since we can choose $\tilde{\varepsilon}, \varepsilon$ and $\tilde{\epsilon}$ arbitrarily small we indeed have the desired result. $\square$

Define

$$Z^n_\infty := \frac{1}{1-\beta^{-1}} \sum_{j=1}^{N} \beta^{H_j^+ - \overline{H}} E[\mathcal{R}^j_\infty] \sum_{k=1}^{B^j_\infty} e_{j,k} \tag{4.48}$$

whose distribution depends on n only through N and $(H_j^+ - \overline{H})_{j=1}^N$. Recalling the definition of $\zeta^{(n)}$ in (4.43), since $e_{j,k}$ are the exponential random variables defining $G^{j,k}$, $B^j_\infty \sim Bin(B^j, p_\infty/p_1(H_1^+))$ and the random variable N is the same in both equations, we have that $\zeta^{(n)}$ and Z^n_∞ are defined on the same probability space.

Proposition 4.5.10. *In IVIE, for any $K \in \mathbb{Z}$ and $\epsilon > 0$*

$$\lim_{n\to\infty} \mathbb{P}^K \left(|\zeta^{(n)} - Z^n_\infty| > \epsilon \right) = 0.$$

Proof. Using the bounds on A_{11}, A_{13} and A_{14} from (4.47) and Lemmas 4.5.8 and 4.5.9 respectively there exists a function $g : \mathbb{R} \to \mathbb{R}$ such that $\lim_{\tilde{\varepsilon}\to 0^+} g(\tilde{\varepsilon}) = 0$ and for sufficiently large n (independently of K)

$$\mathbb{P}^K\left(|\zeta^{(n)} - Z_\infty^n| > \epsilon\right) \leq o(1) + 2\mathbb{P}^K\left(g(\tilde{\varepsilon})Z_\infty^n > \epsilon\right).$$

It therefore suffices to show that $(Z_\infty^n)_{n\geq 0}$ are tight under $\mathbb{P}^K$. Write

$$\mathcal{S}_j := \frac{1}{1-\beta^{-1}} E[\mathcal{R}_\infty^j] \sum_{k=1}^{B_\infty^j} e_{j,k}. \tag{4.49}$$

The variables $E[\mathcal{R}_\infty^j]$, B_∞^j and $e_{j,k}$ are independent, do not depend on K and have finite mean (by Lemma 4.5.1, the geometric distribution of W^j and exponential distribution of $e^{j,k}$) therefore

$$\mathbb{E}^K[\mathcal{S}_j] \leq C < \infty \tag{4.50}$$

uniformly over K. We can then write

$$Z_\infty^n = \sum_{j=1}^{N} \beta^{H_j^+ - \overline{H}_n^K} \mathcal{S}_j.$$

The distribution of $\mathcal{S}_j$ is independent of the height of the trap. The number of large traps N is dominated by the total number of traps $\xi^* - 1$ in the branch thus reintroducing small traps we have that

$$\mathbb{P}^K(Z_\infty^n \geq t) \leq \mathbb{P}^K\left(\sum_{j=1}^{b_n^K \log(t)} \beta^{H_j - \overline{H}_n^K} \mathcal{S}_j \geq t\right) + \mathbb{P}^K(\xi^* - 1 \geq b_n^K \log(t)) \tag{4.51}$$

where we recall that, under $\mathbb{P}^K$, $(H_j)_{j=1}^{\xi^*-1}$ are distributed as the heights of independent f-GW-trees conditioned so that the largest is of height $\overline{H}_n^K$ and $(\mathcal{S}_j)_{j=1}^{\xi^*-1}$ are i.i.d. with the law of $\mathcal{S}_1$. By Lemma 4.5.5 we have that $\lim_{t\to\infty}\limsup_{n\to\infty}\mathbb{P}^K(\xi^* - 1 \geq b_n^K \log(t)) = 0$ therefore it remains to bound the first term in (4.51).

Write $\Phi = \inf\{r \geq 1 : H_r = \overline{H}_n^K\}$ to be the index of the first trap with height the same as the maximum in the branch. Conditional on trap j being the first in the branch which attains the maximum height we have that the heights of the remaining traps are independent and either at most the height of the largest (for higher indices than j) or strictly shorter (for lower indices than j). In particular, this means that

$$\mathbb{P}^K\left(\sum_{j=1}^{b_n^K \log(t)} \beta^{H_j - \overline{H}_n^K} \mathcal{S}_j \geq t\right)$$

$$\leq \mathbb{P}^K\left(\sum_{j=1}^{b_n^K\log(t)}\beta^{H_j-\overline{H}_n^K}\mathcal{S}_j\geq t\Big|\Phi=1\right)$$

$$\leq \mathbb{P}(\mathcal{S}_1\geq\log(t))+\mathbb{P}\left(\sum_{j=2}^{b_n^K\log(t)}\beta^{H_j-\overline{H}_n^K}\mathcal{S}_j\geq t-\log(t)\Big|\bigcap_{j\geq 2}\{H_j\leq\overline{H}_n^K\}\right).$$

The distribution of $\mathcal{S}_1$ is independent of n therefore $\lim_{t\to\infty}\mathbb{P}(\mathcal{S}_1\geq\log(t))=0$. Conditional on $\Phi=1$, $(H_j)_{j\geq 2}$ are independent therefore by Markov's inequality we have that

$$\mathbb{P}\left(\sum_{j=2}^{b_n^K\log(t)}\beta^{H_j-\overline{H}_n^K}\mathcal{S}_j\geq t-\log(t)\Big|\bigcap_{j\geq 2}\{H_j\leq\overline{H}_n^K\}\right)\leq\frac{Cb_n^K\log(t)\mathbb{E}[\beta^{H_1}|H_1\leq\overline{H}_n^K]}{\beta^{\overline{H}_n^K}(t-\log(t))}.$$

For large enough n we have that $\mathbb{P}(H_1\leq\overline{H}_n^K)\geq 1/2$ therefore we have that

$$\mathbb{P}(H_1=l|H_1\leq\overline{H}_n^K)\leq\mathbb{P}(H_1\geq l|H_1\leq\overline{H}_n^K)\leq\frac{1}{2}\mathbb{P}(H_1\geq l)\leq C\mu^l$$

for some constant C therefore the result follows from

$$\mathbb{E}[\beta^{H_1}|H_1\leq\overline{H}_n^K]\ =\ \sum_{l=0}^{\overline{H}_n^K}\beta^l\mathbb{P}(H_1=l|H_1\leq\overline{H}_n^K)\ \leq\ C(\beta\mu)^{\overline{H}_n^K}. \tag{4.52}$$

□

We now prove three technical lemmas which will be important in the proof of Proposition 4.5.14 which is the main result of the section. The first shows that we can reintroduce the small traps into Z_∞^n. The reason for doing this is that we no longer need to condition on the heights of the traps being at least the critical level which will simplify later calculations. In particular, we can replace N with ξ^*-1 (i.e. the total number of traps in the branch) which we understand under $\mathbf{P}^K$ by Lemma 4.5.5.

Lemma 4.5.11. *For all $\tilde{\varepsilon}>0$ we have that for any $K\in\mathbb{Z}$ as $n\to\infty$,*

$$\mathbb{P}^K\left(\sum_{j=1}^{\xi^*-1-N}\beta^{H_j^- -\overline{H}_n^K}\mathcal{S}_j>\tilde{\varepsilon}\right)\to 0.$$

Proof. First, notice that each term in the sum is nonnegative therefore introducing extra terms only increases the probability. By Lemma 4.5.5, for any $\tilde{\epsilon}>0$, we have that $\mathbb{P}^K(\xi^*-1\geq a_{n^{1+\tilde{\epsilon}}})\to 0$ as $n\to\infty$. We therefore have that

$$\mathbb{P}^K\left(\sum_{j=1}^{\xi^*-1-N}\beta^{H_j^- -\overline{H}_n^K}\mathcal{S}_j>\tilde{\varepsilon}\right)\leq\mathbb{P}\left(\sum_{j=1}^{a_{n^{1+\tilde{\epsilon}}}}\beta^{H_j-\overline{H}_n^K}\mathcal{S}_j>\tilde{\varepsilon}\Big|H_j<h_{n,\varepsilon}\ \forall j\geq 1\right)+o(1).$$

By Definitions 4.2.3 and 4.5.4 we have that $\beta^{\overline{H}_n^K} \leq \beta^K a_n^{1/\gamma}$ therefore by Markov's inequality and (4.52) we have that

$$\begin{aligned}\mathbb{P}\left(\sum_{j=1}^{a_{n^{1+\tilde{\epsilon}}}} \beta^{H_j - \overline{H}_n^K} \mathcal{S}_j > \tilde{\varepsilon} \middle| H_j < h_{n,\varepsilon} \ \forall j \geq 1\right) &\leq \frac{a_{n^{1+\tilde{\epsilon}}}\mathbb{E}[\mathcal{S}_1]\mathbb{E}[\beta^{H_1}|H_1 < h_{n,\varepsilon}]}{\tilde{\varepsilon}\beta^{\overline{H}_n^K}} \\ &\leq \frac{C_{K,\tilde{\varepsilon}} a_{n^{1+\tilde{\epsilon}}} (\beta\mu)^{h_{n,\varepsilon}}}{a_n^{1/\gamma}}.\end{aligned}$$

Recall from Definition 4.2.3 that $h_{n,\varepsilon} \leq \log(a_{n^{1-\varepsilon}})/\log(\mu^{-1})$ therefore

$$(\beta\mu)^{h_{n,\varepsilon}} \leq a_{n^{1-\varepsilon}}^{\frac{1}{\gamma}-1}.$$

Using the form of a_n following Definition 4.0.3 we then have that there exists a slowly varying function $\overline{L}$ such that

$$\frac{a_{n^{1+\tilde{\epsilon}}}(\beta\mu)^{h_{n,\varepsilon}}}{a_n^{1/\gamma}} \leq \overline{L}(n) n^{\frac{1}{\alpha-1}\left(\tilde{\epsilon}+\varepsilon-\frac{\varepsilon}{\gamma}\right)}$$

which converges to 0 by choosing $\tilde{\epsilon} < \varepsilon(1/\gamma - 1)$. □

The second Lemma leading to Proposition 4.5.14 shows that the height of an f-GW-tree is sufficiently close to a geometric random variable. To ease notation let $\mathcal{S} = \mathcal{S}_1$ (see (4.49)), $H = H_1 \sim \mathcal{H}(\mathcal{T}^f)$ be distributed as the height of a GW-tree and $G \sim Geo(\mu)$ independently of each other.

Lemma 4.5.12. *In IVIE,*

$$b\left|\int_0^\infty e^{-x}\mathbb{P}\left(\mathcal{S}\beta^H \geq \frac{xb^{1/\gamma}}{\theta}\right)\mathrm{d}x - c_\mu \int_0^\infty e^{-x}\mathbb{P}\left(\mathcal{S}\beta^G \geq \frac{xb^{1/\gamma}}{\theta}\right)\mathrm{d}x\right| \tag{4.53}$$

converges to zero as $b \to \infty$.

Proof. From (1.2) and (4.50) we have that $\gamma < 1$ and $\mathbb{E}[\mathcal{S}] < \infty$ therefore $\mathbb{E}[\mathcal{S}^\gamma] < \infty$. By independence of $\mathcal{S}$ and G

$$\mathbb{P}\left(\mathcal{S}\beta^G \geq \frac{xb^{1/\gamma}}{\theta}\right) = \mathbb{E}\left[\mathbb{P}\left(G \geq \frac{\log(xb^{1/\gamma}(\mathcal{S}\theta)^{-1})}{\log(\beta)}\middle|\mathcal{S}\right)\right] \leq \left(\frac{xb^{1/\gamma}}{\theta}\right)^{-\gamma}\mathbb{E}[\mathcal{S}^\gamma] = \frac{C_\theta}{bx^\gamma}.$$

Similarly, since there exists a constant c such that $\mathbb{P}(H \geq t) \leq c\mathbb{P}(G \geq t)$ uniformly over t we have that

$$\mathbb{P}\left(\mathcal{S}\beta^H \geq \frac{xb^{1/\gamma}}{\theta}\right) \leq \frac{C_\theta}{bx^\gamma}.$$

Let $\tilde{\varepsilon} > 0$ then choose $\epsilon > 0$ such that

$$\int_0^\epsilon e^{-x} x^{-\gamma} \mathrm{d}x < \frac{\tilde{\varepsilon}}{C_\theta}$$

then, since the integrals are positive and $c_\mu \leq 1$, we have that

$$\left| \int_0^\epsilon e^{-x} c_\mu \mathbb{P}\left(\mathcal{S}\beta^G \geq \frac{xb^{1/\gamma}}{\theta} \right) \mathrm{d}x - \int_0^\epsilon e^{-x} \mathbb{P}\left(\mathcal{S}\beta^H \geq \frac{xb^{1/\gamma}}{\theta} \right) \mathrm{d}x \right| \leq \tilde{\varepsilon} b^{-1}. \tag{4.54}$$

By (2.9) we have that

$$m(b) := \sup_{z > \frac{\epsilon b}{\theta}} \left| \frac{\mathbb{P}\left(\beta^H \geq z\right)}{\mathbb{P}\left(\beta^G \geq z\right)} - c_\mu \right| \tag{4.55}$$

converges to 0 as $b \to \infty$. Now define $M(b) := m(b)^{1-\frac{1}{\gamma}} \wedge b^{\frac{1}{\gamma}-1}$ then $M(b) \to \infty$ as $b \to \infty$ but $M(b) \ll b^{1/\gamma}$.

For $x > \epsilon$, by independence of $\mathcal{S}$ and H we have that

$$\begin{aligned} \mathbb{P}\left(\mathcal{S}\beta^H \geq \frac{xb^{1/\gamma}}{\theta} \Big| \mathcal{S} \geq M(b) \right) \; &\leq \; C\mathbb{E}\left[\left(\frac{xb^{1/\gamma}}{\theta \mathcal{S}} \right)^{\frac{\log(\mu)}{\log(\beta)}} \Big| \mathcal{S} \geq M(b) \right] \\ &\leq \; C_{\epsilon,\theta} b^{-1} \mathbb{E}\left[\mathcal{S}^\gamma | \mathcal{S} \geq M(b) \right]. \end{aligned}$$

In particular,

$$b\mathbb{P}\left(\mathcal{S}\beta^H \geq \frac{xb^{1/\gamma}}{\theta},\; \mathcal{S} \geq M(b) \right) \leq C_{\epsilon,\theta} \mathbb{E}\left[\mathcal{S}^\gamma \mathbf{1}_{\{\mathcal{S} \geq M(b)\}} \right]$$

which converges to 0 as $b \to \infty$ by dominated convergence. Similarly, the same holds replacing H with G therefore combining this with (4.54) we have that the quantity (4.53) is bounded above by

$$\tilde{\varepsilon} + o(1) + Cb \sup_{x > \epsilon} \left| \mathbb{P}\left(\mathcal{S}\beta^H \geq \frac{xb^{1/\gamma}}{\theta},\; \mathcal{S} < M(b) \right) - c_\mu \mathbb{P}\left(\mathcal{S}\beta^G \geq \frac{xb^{1/\gamma}}{\theta},\; \mathcal{S} < M(b) \right) \right|.$$

Since $\mathcal{S}$ is independent of G and H we have that the supremum in the above expression can be bounded above by

$$\sup_{z > \frac{\epsilon b^{1/\gamma}}{\theta M(b)}} \left| \mathbb{P}\left(\beta^H \geq z\right) - c_\mu \mathbb{P}\left(\beta^G \geq z\right) \right| \leq m(b) \mathbb{P}\left(G \geq \frac{\log(\epsilon b^{1/\gamma}(\theta M(b))^{-1})}{\log(\beta)} \right)$$

by (4.55) since $b^{1/\gamma}/M(b) \geq b$. Since $G \sim Geo(\mu)$ we have that

$$m(b)\mathbb{P}\left(G \geq \frac{\log(\epsilon b^{1/\gamma}(\theta M(b))^{-1})}{\log(\beta)}\right) = C_{\epsilon,\theta} m(b)\left(\frac{b^{1/\gamma}}{M(b)}\right)^{\frac{\log(\mu)}{\log(\beta)}} \leq \frac{C_{\epsilon,\theta} m(b)^\gamma}{b}$$

which completes the proof. $\square$

In the final Lemma preceding Proposition 4.5.14 we show that the Laplace transform

$$\varphi_K(\lambda) := \mathbb{E}^K\left[e^{-\lambda\sum_{j=1}^{\xi^*-1}\beta^{H_j-\overline{H}_n^K}\mathcal{S}_j}\right]$$

can be written in terms of the distributions of $\mathcal{S}, H$ and ξ^*.

Lemma 4.5.13. *In IVIE,*

$$\varphi_K(\lambda) = \mathbb{E}^K\left[\frac{\mathbb{E}\left[e^{-\lambda\mathcal{S}\beta^{H-\overline{H}_n^K}}\mathbf{1}_{\{H\leq\overline{H}_n^K\}}\right]^{\xi^*-1} - \mathbb{E}\left[e^{-\lambda\mathcal{S}\beta^{H-\overline{H}_n^K}}\mathbf{1}_{\{H\leq\overline{H}_n^K-1\}}\right]^{\xi^*-1}}{\mathbf{P}(H\leq\overline{H}_n^K)^{\xi^*-1} - \mathbf{P}(H\leq\overline{H}_n^K-1)^{\xi^*-1}}\right]$$

Proof. Recall that $\Phi := \inf\{r \geq 1 : H_r = \overline{H}\}$ is the index of the first random variable in the sequence $(H_j)_{j=1}^{\xi^*-1}$ which attains the maximum value $\overline{H} := \max_{j\leq\xi^*-1} H_j$. For $h \in \mathbb{Z}^+, \lambda > 0$ and $i = 1, 2$ write

$$\begin{aligned}
\psi_i(h,\lambda) &:= \mathbb{E}\left[e^{-\lambda\mathcal{S}\beta^{H-h}}|H\leq h+1-i\right], \\
\phi_i(h,\lambda) &:= \mathbb{E}\left[e^{-\lambda\mathcal{S}\beta^{H-h}}\mathbf{1}_{\{H\leq h+1-i\}}\right] = \psi_i(h,\lambda)\mathbb{P}(H\leq h+1-i).
\end{aligned} \tag{4.56}$$

Conditional on Φ, the random variables $(H_j)_{j\geq 1}$ are independent with

$$\mathbb{P}^K\left(H_j = z|\Phi\right) = \begin{cases} \mathbf{1}_{\{z=\overline{H}_n^K\}}, & \text{if } j = \Phi, \\ \mathbb{P}(H = z|H\leq\overline{H}_n^K - 1), & \text{if } j < \Phi, \\ \mathbb{P}(H = z|H\leq\overline{H}_n^K), & \text{if } j > \Phi. \end{cases}$$

By conditioning on ξ^*, we then have that

$$\begin{aligned}
\varphi_K(\lambda) &= \mathbb{E}^K\left[\sum_{k=1}^{\xi^*-1}\mathbf{P}^K(\Phi = k|\xi^*)\mathbb{E}^K\left[e^{-\lambda\sum_{j=1}^{\xi^*-1}\beta^{H_j-\overline{H}_n^K}\mathcal{S}_j}\Big|\Phi = k,\xi^*\right]\right] \\
&= \mathbb{E}^K\left[\mathbb{E}[e^{-\lambda\mathcal{S}}]\sum_{k=1}^{\xi^*-1}\mathbf{P}^K(\Phi = k|\xi^*)\psi_2(\overline{H}_n^K,\lambda)^{k-1}\psi_1(\overline{H}_n^K,\lambda)^{\xi^*-1-k}\right]
\end{aligned} \tag{4.57}$$

and by Bayes' rule we also have that

$$\mathbf{P}^K(\Phi = k|\xi^*) = \frac{\mathbf{P}(H = \overline{H}_n^K)}{\mathbf{P}(\overline{H} = \overline{H}_n^K|\xi^*)}\mathbf{P}(H \leq \overline{H}_n^K - 1)^{k-1}\mathbf{P}(H \leq \overline{H}_n^K)^{\xi^*-1-k}. \tag{4.58}$$

Combining (4.56), (4.57) and (4.58) we can then write $\varphi_K(\lambda)$ as

$$\mathbb{E}^K\left[\frac{\mathbb{E}[e^{-\lambda\mathcal{S}}]\mathbf{P}(H = \overline{H}_n^K)}{\mathbf{P}(\overline{H} = \overline{H}_n^K|\xi^*)}\sum_{k=1}^{\xi^*-1}\phi_2(\overline{H}_n^K,\lambda)^{k-1}\phi_1(\overline{H}_n^K,\lambda)^{\xi^*-1-k}\right]. \tag{4.59}$$

For $0 < p < q < 1$ and $l \in \mathbb{Z}^+$,

$$\sum_{k=1}^{l} p^{k-1}q^{l-k} \;=\; q^{l-1}\sum_{k=0}^{l-1}\left(\frac{p}{q}\right)^k \;=\; q^{l-1}\left(\frac{1-\left(\frac{p}{q}\right)^l}{1-\frac{p}{q}}\right) \;=\; \frac{q^l-p^l}{q-p}.$$

Since $0 < \phi_2(\overline{H}_n^K,\lambda)^{k-1} < \phi_1(\overline{H}_n^K,\lambda)^{k-1} < 1$, by (4.59) it follows that that $\varphi_K(\lambda)$ is equal to

$$\mathbb{E}^K\left[\frac{\mathbb{E}[e^{-\lambda\mathcal{S}}]\mathbf{P}(H = \overline{H}_n^K)}{\mathbf{P}(\overline{H} = \overline{H}_n^K|\xi^*)}\left(\frac{\phi_1(\overline{H}_n^K,\lambda)^{\xi^*-1}-\phi_2(\overline{H}_n^K,\lambda)^{\xi^*-1}}{\phi_1(\overline{H}_n^K,\lambda)-\phi_2(\overline{H}_n^K,\lambda)}\right)\right]$$

however, from (4.56),

$$\phi_1(\overline{H}_n^K,\lambda)-\phi_2(\overline{H}_n^K,\lambda) = \mathbb{E}[e^{-\lambda\mathcal{S}}]\mathbf{P}(H = \overline{H}_n^K)$$

therefore this is equal to

$$\mathbb{E}^K\left[\frac{\phi_1(\overline{H}_n^K,\lambda)^{\xi^*-1}-\phi_2(\overline{H}_n^K,\lambda)^{\xi^*-1}}{\mathbf{P}(\overline{H} = \overline{H}_n^K|\xi^*)}\right].$$

The result then follows from

$$\begin{aligned}\mathbf{P}(\overline{H} = \overline{H}_n^K|\xi^*) \;&=\; \mathbf{P}(\overline{H} \leq \overline{H}_n^K|\xi^*) - \mathbf{P}(\overline{H} \leq \overline{H}_n^K - 1|\xi^*)\\ &=\; \mathbf{P}(H \leq \overline{H}_n^K)^{\xi^*-1} - \mathbf{P}(H \leq \overline{H}_n^K - 1)^{\xi^*-1}\end{aligned}$$

which is a consequence of $\overline{H}$ being the maximum of ξ^*-1 i.i.d. random variables. □

The next proposition shows that, under $\mathbb{P}^K$, we have that the scaled time spent in a large branch $\zeta^{(n)}$ (from (4.43)) converges in distribution along subsequences n_l where $a_{n_l(t)} \sim t\mu^{-l}$.

Proposition 4.5.14. *In IVIE, under $\mathbb{P}^K$ we have that $Z_\infty^{n_l}$ converges in distribution (as $l \to \infty$) to a random variable Z_∞.*

Proof. By Lemmas 4.5.11 and 4.5.13, it now suffices to show convergence of

$$\varphi_K(\lambda) = \mathbb{E}^K\left[\frac{\mathbb{E}\left[e^{-\lambda\mathcal{S}\beta^{H-\overline{H}_n^K}}\mathbf{1}_{\{H\le\overline{H}_n^K\}}\right]^{\xi^*-1} - \mathbb{E}\left[e^{-\lambda\mathcal{S}\beta^{H-\overline{H}_n^K}}\mathbf{1}_{\{H\le\overline{H}_n^K-1\}}\right]^{\xi^*-1}}{\mathbf{P}(H\le\overline{H}_n^K)^{\xi^*-1} - \mathbf{P}(H\le\overline{H}_n^K-1)^{\xi^*-1}}\right]. \tag{4.60}$$

By (4.2) we have that $\mathbf{P}(H\le\overline{H}_n^K) = 1 - c_\mu\mu^{1+\overline{H}_n^K}(1+o(1))$ therefore, using the relationship (4.40) between b_n^K and $\overline{H}_n^K$ we have that

$$\mathbf{P}(H\le\overline{H}_n^K)^{\xi^*-1} = \left(1-\frac{\mu(1+o(1))}{b_n^K}\right)^{\xi^*-1} = \exp\left(-\frac{\xi^*-1}{b_n^K}\mu(1+o(1))\right)$$

and similarly,

$$\mathbf{P}(H\le\overline{H}_n^K-1)^{\xi^*-1} = \exp\left(-\frac{\xi^*-1}{b_n^K}(1+o(1))\right).$$

By Lemma 4.5.5 we know that $(\xi^*-1)/b_n^K$ converges in distribution to a random variable with exponential moments therefore we want to show a similar expression for the numerator in (4.60). Notice that

$$\mathbb{E}\left[e^{-\lambda\mathcal{S}\beta^{H-\overline{H}_n^K}}\mathbf{1}_{\{H\le\overline{H}_n^K\}}\right] = \mathbb{E}\left[e^{-\lambda\mathcal{S}\beta^{H-\overline{H}_n^K}}\right]\left(1-\frac{\mathbb{E}\left[e^{-\lambda\mathcal{S}\beta^{H-\overline{H}_n^K}}\mathbf{1}_{\{H>\overline{H}_n^K\}}\right]}{\mathbb{E}\left[e^{-\lambda\mathcal{S}\beta^{H-\overline{H}_n^K}}\right]}\right) \tag{4.61}$$

where $\mathbb{E}\left[e^{-\lambda\mathcal{S}\beta^{H-\overline{H}_n^K}}\right]$ converges to 1 deterministically. In particular, this means that

$$\begin{aligned}&\left(1-\frac{\mathbb{E}\left[e^{-\lambda\mathcal{S}\beta^{H-\overline{H}_n^K}}\mathbf{1}_{\{H>\overline{H}_n^K\}}\right]}{\mathbb{E}\left[e^{-\lambda\mathcal{S}\beta^{H-\overline{H}_n^K}}\right]}\right)^{\xi^*-1}\\ &\qquad = \exp\left(-(\xi^*-1)\mathbb{E}\left[e^{-\lambda\mathcal{S}\beta^{H-\overline{H}_n^K}}\mathbf{1}_{\{H>\overline{H}_n^K\}}\right](1+o(1))\right).\end{aligned} \tag{4.62}$$

By summing over the possible values of H and using independence of $\mathcal{S}$ and H we have that

$$\mathbb{E}\left[e^{-\lambda\mathcal{S}\beta^{H-\overline{H}_n^K}}\mathbf{1}_{\{H>\overline{H}_n^K\}}\right] = (1+o(1))\sum_{j=\overline{H}_n^K+1}^{\infty}(1-\mu)\mu^j\mathbb{E}\left[e^{-\lambda\mathcal{S}\beta^{j-\overline{H}_n^K}}\right]$$

$$= (1+o(1))\mu^{\overline{H}_n^K+1}\sum_{j=0}^{\infty}(1-\mu)\mu^j\mathbb{E}\left[e^{-\lambda\mathcal{S}\beta^{j+1}}\right].$$

Recalling $G \sim Geo(\mu)$ independently of $\mathcal{S}$ then writing $\varphi^{SG}(\lambda)$ to be the Laplace transform of $\mathcal{S}\beta^G$ and using the relationship (4.40) between b_n^K and $\overline{H}_n^K$ we therefore have that (4.62) can be written as

$$\exp\left(-\frac{\xi^*-1}{b_n^K}\mu\varphi^{SH}(\lambda\beta)(1+o(1))\right). \tag{4.63}$$

It remains to deal with $\mathbb{E}[e^{-\lambda\mathcal{S}\beta^{H-\overline{H}_n^K}}]^{b_n^K}$. To ease notation, let us write $b := b_n^K = c_\mu^{-1}\mu^{-\overline{H}_n^K}$ and $\theta = \lambda c_\mu^{-1/\gamma}$ then

$$\begin{aligned}
\mathbb{E}\left[e^{-\lambda\mathcal{S}\beta^{H-\overline{H}_n^K}}\right]^{b_n^K} &= \mathbb{E}\left[e^{-\theta\mathcal{S}\beta^H b^{-1/\gamma}}\right]^b \\
&= \left(\int_0^1 \mathbb{P}\left(e^{-\theta\mathcal{S}\beta^H b^{-1/\gamma}} \geq y\right)\mathrm{d}y\right)^b \\
&= \left(1-\int_0^1 \mathbb{P}\left(\mathcal{S}\beta^H \geq -\frac{\log(y)b^{1/\gamma}}{\theta}\right)\mathrm{d}y\right)^b \\
&= \left(1-\int_0^\infty e^{-x}\mathbb{P}\left(\mathcal{S}\beta^H \geq \frac{xb^{1/\gamma}}{\theta}\right)\mathrm{d}x\right)^b \\
&= \left(1-\int_0^\infty e^{-x}c_\mu\mathbb{P}\left(\mathcal{S}\beta^G \geq \frac{xb^{1/\gamma}}{\theta}\right)\mathrm{d}x\right)^b + o(1) \qquad (4.64)
\end{aligned}$$

where the final equality holds by Lemma 4.5.12. Since $\mathcal{S}$ and G are independent we have that

$$\mathbb{P}\left(\mathcal{S}\beta^G \geq z\right) \;=\; \mathbb{E}\left[\mathbb{P}\left(G \geq \frac{\log(z/\mathcal{S})}{\log(\beta)}\Big|\mathcal{S}\right)\right] \;=\; \mathbb{E}\left[\mu^{\left\lceil\frac{\log(z/\mathcal{S})}{\log(\beta)}\right\rceil}\right].$$

Writing

$$J(z) := \left\lceil\frac{\log(z)}{\log(\beta)}\right\rceil - \frac{\log(z)}{\log(\beta)} \qquad \text{and} \qquad I(z) := \mathbb{E}\left[\mathcal{S}^\gamma\mu^{-\left\lfloor\frac{\log(\mathcal{S})}{\log(\beta)}+J(z)\right\rfloor+\frac{\log(\mathcal{S})}{\log(\beta)}+J(z)}\right]$$

we then have that

$$\mathbb{P}(\mathcal{S}\beta^G \geq z) \;=\; \mu^{\frac{\log(z)}{\log(\beta)}}\mathbb{E}\left[\mathcal{S}^\gamma\mu^{-\left\lfloor\frac{\log(\mathcal{S})}{\log(\beta)}+J(z)\right\rfloor+\frac{\log(\mathcal{S})}{\log(\beta)}+J(z)}\right] \;=\; z^{-\gamma}I(z)$$

where, from (4.50), we also have that $I(z) \leq \mathbb{E}[\mathcal{S}^\gamma] < \infty$ since $\gamma < 1$ by (1.2). Moreover, $J(z) = J(zm^{\log(\beta)})$ and $I(z) = I(zm^{\log(\beta)})$ for all $z \in \mathbb{R}$, $m \in \mathbb{Z}$.

Substituting this back into (4.64) we have that

$$\mathbb{E}\left[e^{-\theta\mathcal{S}\beta^{G}b^{-1/\gamma}}\right]^{b} = \left(1-\theta^{\gamma}b^{-1}\int_0^\infty e^{-x}x^{-\gamma}I\left(\frac{xb^{1/\gamma}}{\theta}\right)\mathrm{d}x\right)^{b}+o(1).$$

For $t>0$, along sequences $n_l(t)$ such that $a_{n_l(t)}\sim t\mu^{-l}$ we have that $(b_n^K)^{1/\gamma}\sim Cl^{\log(\beta)}$ therefore, since I is bounded, we have that along subsequences $n_l(t)$

$$\int_0^\infty e^{-x}x^{-\gamma}I\left(\frac{xb^{1/\gamma}}{\theta}\right)\mathrm{d}x$$

converges to some positive function of θ. In particular, we have that

$$\mathbb{E}\left[e^{-\lambda\mathcal{S}\beta^{H-\overline{H}_n^K}}\right]^{b_n^K}$$

converges to a constant in the interval $(0,1)$. Combining this with (4.61) and (4.63) we have that

$$\mathbb{E}\left[e^{-\lambda\mathcal{S}\beta^{H-\overline{H}_n^K}}\mathbf{1}_{\{H\leq\overline{H}_n^K\}}\right]^{\xi^*-1} = \exp\left(-\frac{\xi^*-1}{b_n^K}\mu C_{\lambda,\beta}(1+o(1))\right)$$

for some constant $C_{\mu,\beta}$ depending on the distribution of $\mathcal{S}$. Furthermore, the same arguments gives us that

$$\mathbb{E}\left[e^{-\lambda\mathcal{S}\beta^{H-\overline{H}_n^K}}\mathbf{1}_{\{H\leq\overline{H}_n^K-1\}}\right]^{\xi^*-1} = \exp\left(-\frac{\xi^*-1}{b_n^K}C_{\lambda,\beta}(1+o(1))\right).$$

By boundedness, continuity and Lemma 4.5.5 we therefore have that $\varphi_K(\lambda)$ converges along the given subsequences which proves the result. $\square$

In order to prove the convergence result for sums of i.i.d. variables we will require that $\zeta^{(n)}$ can be dominated (independently of $K\geq h_{n,\varepsilon}-h_{n,0}$) by a random variable Z_{sup} such that $\mathbb{E}[Z_{sup}^{(\alpha-1)\gamma+\epsilon}]<\infty$ for ϵ sufficiently small. Lemma 4.5.15 shows that we indeed have the domination required.

Lemma 4.5.15. *In IVIE, there exists a random variable Z_{sup} such that under $\mathbb{P}^K$ for any $K\in\mathbb{Z}$ we have that $Z_{sup}\succeq\zeta^{(n)}$ for all n sufficiently large and $\mathbb{E}[Z_{sup}^{1-\epsilon}]<\infty$ for any $\epsilon>0$.*

Proof. The number of large traps N is dominated by the number of traps in the branch. Similarly to Lemma 4.5.5 we consider

$$\mathbf{P}(\xi^*-1\geq tb_n^K|\overline{H}=\overline{H}_n^K)=\mathbf{P}(\overline{H}=\overline{H}_n^K|\xi^*-1\geq tb_n^K)\frac{\mathbf{P}(\xi^*-1\geq tb_n^K)}{\mathbf{P}(\overline{H}=\overline{H}_n^K)}.$$

Using the tail of H from (2.9), for large n (independently of $t \geq 0$) and some constant c, we can bound $\mathbf{P}(\overline{H} = \overline{H}_n^K | \xi^* - 1 \geq t b_n^K)$ above by

$$\mathbf{E}\left[\mathbf{P}(H \leq \overline{H}_n^K)^{\xi^*-1}\middle|\xi^* - 1 \geq t b_n^K\right] \;\leq\; \mathbf{E}\left[e^{-c\left(\frac{\xi^*-1}{b_n^K}\right)}\middle|\xi^* - 1 \geq t b_n^K\right] \;\leq\; e^{-ct}.$$

For each $t \geq 0$ we have that $\mathbf{P}(\xi^* - 1 \geq t b_n^K) \sim C t^{-(\alpha-1)} \mathbf{P}(\overline{H} = \overline{H}_n^K)$ as $n \to \infty$. Since $\mathbf{P}(\overline{H} = \overline{H}_n^K)$ does not depend on t we can choose a constant c such that for n sufficiently large we have that $\mathbf{P}(\overline{H} = \overline{H}_n^K) \leq c\mathbf{P}(\xi^* - 1 \geq b_n^K)$ thus for $t \geq 1$

$$\frac{\mathbf{P}(\xi^* - 1 \geq t b_n^K)}{\mathbf{P}(\overline{H} = \overline{H}_n^K)} \leq \frac{\mathbf{P}(\xi^* - 1 \geq t b_n^K)}{c\mathbf{P}(\xi^* - 1 \geq b_n^K)} \leq c^{-1}.$$

In particular, for $t \geq 1$ we have that $\mathbf{P}(\xi^* - 1 \geq t b_n^K | \overline{H} = \overline{H}_n^K) \leq c_1 e^{-c_2 t}$ for some constants c_1, c_2. It follows that there exists some random variable ξ_{sup} which is independent of $\overline{H}$, has an exponential tail and satisfies $\xi_{sup} b_n^K \geq \xi^* - 1$ on the event $\{\overline{H} = \overline{H}_n^K\}$ for n suitably large (independently of K).

Recall that the total number of excursions W^j in a trap exceeds the number B^j which reach the apex and we write $G^{j,k}$ to denote the number of excursions from the apex. The length of these excursions can be dominated by excursions $\mathcal{R}_\infty^{j,k,l}$ from the apexes of the infinite traps $\mathcal{T}_i^{\prec}$. We then have that for n suitably large, under $\mathbb{P}^K$

$$\zeta^{(n)} \preceq \sum_{j=1}^{\xi_{sup} b_n^K} \sum_{k=1}^{W^j} \sum_{l=1}^{G^{j,k}} \frac{\mathcal{R}_\infty^{j,k,l}}{\beta^{\overline{H}_n^K}}.$$

By (4.45) $E[G^{j,k}] \leq \beta^{H_j+1}/(\beta+1)$ therefore there is some constant c such that, writing

$$\mathscr{Y}_j^{(n)} := c \sum_{k=1}^{W^j} \sum_{l=1}^{G^{j,k}} \frac{\mathcal{R}_\infty^{j,k,l}}{E[G^{j,k}]}$$

(which are identically distributed under $\mathbb{P}$) we have that under $\mathbb{P}^K$,

$$\zeta^{(n)} \preceq \frac{1}{\beta^{\overline{H}_n^K}} \sum_{j=1}^{\xi_{sup} b_n^K} \beta^{H_j} \mathscr{Y}_j^{(n)}.$$

For $m \geq 1$ write $\mathscr{X}^n(m) := \frac{1}{m}\sum_{j=1}^m \beta^{H_j} \mathscr{Y}_j^{(n)} \mathbf{1}_{\{j \neq \Phi\}}$ (where we recall that Φ is the first index j such that $H_j = \overline{H}_n^K$) then by Markov's inequality

$$\mathbb{P}^K(\mathscr{X}^n(m) \geq t) \;\leq\; \frac{1}{m}\sum_{j=1}^m \frac{\mathbb{E}^K\left[\beta^{H_j} \mathscr{Y}_j^{(n)} \mathbf{1}_{\{j\neq\Phi\}}\right]}{t} \;=\; \frac{1}{m}\sum_{j=1}^m \frac{\mathbb{E}^K\left[\beta^{H_j} \mathbf{1}_{\{j\neq\Phi\}}\right] \mathbb{E}^K\left[\mathscr{Y}_j^{(n)}\right]}{t}$$

since $\mathbb{E}[\mathscr{Y}_j^{(n)}|H_j, \Phi]$ is independent of H_j and Φ. Since W^1 has a geometric distribution (independently of n) we have that $\mathbb{E}[W^1] < \infty$ and by Lemma 4.5.1 we have that $\mathbb{E}[\mathcal{R}_\infty] < \infty$ therefore $\mathbb{E}^K[\mathscr{Y}_j^{(n)}] \leq \mathbb{E}[W^1]\mathbb{E}[\mathcal{R}_\infty] \leq C < \infty$ for all n. Using geometric bounds on the tail of H from (2.9) and that $\mathbb{P}(H \geq j | H \leq \overline{H}_n^K) \leq \mathbb{P}(H \geq j)$ we have that

$$\mathbb{E}^K[\beta^H] \;\leq\; \sum_{j=0}^{\infty} \beta^j \mathbb{P}(H \geq j | H \leq \overline{H}_n^K) \;\leq\; C(\beta\mu)^{\overline{H}_n^K}.$$

We therefore have that $\mathbb{P}^K(\mathscr{X}^n(m) \geq t) \leq C(\beta\mu)^{\overline{H}_n^K}/t$ thus there exists some sequence of random variables $\mathscr{X}_{sup}^n \succeq \mathscr{X}^n(m)$ for any m such that $\mathbb{P}^K(\mathscr{X}_{sup}^n \geq t) = 1 \wedge C(\beta\mu)^{\overline{H}_n^K} t^{-1}$. In particular, $\mathscr{X}_{sup}^n \succeq \mathscr{X}^n(\xi_{sup} b_n^K)$. Therefore,

$$\frac{1}{\beta^{\overline{H}_n^K}} \sum_{j=1}^{\xi_{sup} b_n^K} \beta^{H_j} \mathscr{Y}_j^{(n)} = \frac{\xi_{sup} b_n^K}{\beta^{\overline{H}_n^K}} \mathscr{X}^n(\xi_{sup} b_n^K) + \mathscr{Y}_\Phi^{(n)} \preceq \frac{\xi_{sup} \mathscr{X}_{sup}^n}{c_\mu(\beta\mu)^{\overline{H}_n^K}} + \mathscr{Y}_\Phi^{(n)}$$

under $\mathbb{P}^K$. We then have that

$$\mathbb{P}^K\left(\frac{\xi_{sup}\mathscr{X}_{sup}^n}{c_\mu(\beta\mu)^{\overline{H}_n^K}} \geq t\right) = \mathbb{E}^K\left[\mathbb{P}^K\left(\mathscr{X}_{sup}^n \geq \frac{tc_\mu(\beta\mu)^{\overline{H}_n^K}}{\xi_{sup}} \Big| \xi_{sup}\right)\right] = 1 \wedge C\frac{\mathbb{E}^K[\xi_{sup}]}{t}$$

where ξ_{sup} has finite first moment since $\mathbf{P}(\xi_{sup} \geq t) = c_1 e^{-c_2 t} \wedge 1$. It follows that there exists $\mathscr{X}_{sup} \succeq \mathscr{X}_{sup}^n$ for any n such that $\mathbb{P}(\mathscr{X}_{sup} \geq t) = 1 \wedge Ct^{-1}$.

Since $\mathbb{E}^K[\mathscr{Y}_\Phi^n]$ is bounded independently of K and n, by Markov's inequality we have that there exists $\mathscr{Y}_{sup} \succeq \mathscr{Y}_\Phi^n$ for all n such that $\mathbb{P}(\mathscr{Y}_{sup} \geq t) = 1 \wedge Ct^{-1}$. It therefore follows that $\zeta^{(n)}$ under $\mathbb{P}^K$ is stochastically dominated by $\mathscr{X}_{sup} + \mathscr{Y}_{sup}$ under $\mathbb{P}$ where

$$\mathbb{P}(\mathscr{X}_{sup} + \mathscr{Y}_{sup} \geq t) \;\leq\; \mathbb{P}(\mathscr{X}_{sup} \geq t/2) + \mathbb{P}(\mathscr{Y}_{sup} \geq t/2) \;\leq\; Ct^{-1}$$

hence $\mathscr{X}_{sup} + \mathscr{Y}_{sup}$ has finite moments up to $1 - \epsilon$ for all $\epsilon > 0$. $\square$

4.6 Convergence of the random sum along specific subsequences

In this section we prove the main theorems concerning convergence to infinitely divisible laws in FVIE and IVIE. Both cases follow the proof from [10]; in FVIE the result follows directly whereas in IVIE adjustments need to be made to deal with slowly varying functions.

Recall that we want to show convergence of Δ_n/a_n along sequences $n_l(t)$ however, by Corollary 4.3.4 and Lemma 4.3.5, it suffices to consider

$$\tilde{\chi}_{t,n} = \sum_{i=1}^{\lfloor ntq_n \rfloor} \tilde{\chi}_n^i$$

where $\tilde{\chi}_n^i$ is the time spent in large traps of the i^{th} large branch by walk $X_n^{(i)}$. Furthermore, by Proposition 4.5.2 and Corollary 4.5.3 we can replace $\tilde{\chi}_n^i$ with $\tilde{\chi}_n^{i*}$ which is the time spent on excursions from the deepest point of the traps of the i^{th} branch by $X_n^{(i)}$.

Let $\overline{H}_i$ denote the height of the largest trap in the i^{th} large branch then for $i, l \geq 1$ let $\zeta_i^l := \tilde{\chi}_{n_l}^{i*} \beta^{-\overline{H}_i}$ then $(\zeta_i^l)_{i \geq 1}$ are i.i.d. with the law of $\zeta^{(n_l)}$. Let $n_l := n_l(1)$ then for $K \geq -(l - h_{n_l,\varepsilon})$ let $\zeta_i^{l,K}$ be ζ_i^l conditioned on the event $\{\overline{H}_i = l + K\}$ when this makes sense and 0 otherwise. For $K \in \mathbb{Z}$ and $l \geq 0$ define $\overline{F}_K^l(x) := \mathbb{P}(\zeta_i^{l,K} > x)$.

Proof of Theorem 4.2 (FVIE)

Recall that in FVIE $\gamma = \log(\mu^{-1})/\log(\beta) < 1$, $n_l(t) = \lfloor t\mu^{-l} \rfloor$ and by Corollary 4.1.3 we have that the height of a branch decays exponentially: $\mathbf{P}(\mathcal{H}(\mathcal{T}^{*-}) \geq n) \sim C_{\mathcal{D}} \mu^n = C_{\mathcal{D}} \beta^{-n\gamma}$ where $C_{\mathcal{D}} = c_\mu \mathbf{E}[\xi^* - 1]$.

By a simple adaptation of Corollary 4.5.10 and Lemma 4.5.15

1. $\exists Z_\infty^{(i)}$ random variables such that for all $K \in \mathbb{Z}$ we have that $\zeta_i^{l,K} \xrightarrow{\text{d}} Z_\infty^{(i)}$ as $l \to \infty$;

2. $\exists Z_{sup}$ random variable such that for all $l \geq 0$ and $K \geq -(l - h_{n_l,\varepsilon})$ we have that $\zeta_i^{l,K} \preceq Z_{sup}$ and $\mathbf{E}[Z_{sup}^{\gamma+\epsilon}] < \infty$ for some $\epsilon > 0$.

More specifically, since there is precisely one large trap in a large branch with high probability in FVIE the random variable in (4.48) can be written as

$$Z_\infty^n = \frac{1}{1 - \beta^{-1}} E[\mathcal{R}_\infty] \sum_{k=1}^{B_\infty} e_k$$

for some binomial variable B_∞ and independent exponential variables e_k. These are independent of n, hence an adaptation of Proposition 4.5.10 shows that $\zeta^{(n)}$ converge in distribution under $\mathbb{P}^K$.

Set

$$S_M^l := \sum_{i=1}^{M} \tilde{\chi}_{n_l}^{i*}.$$

For $(\lambda_l)_{l \geq 0}$ converging to $\lambda > 0$ define $M_l^\lambda := \lfloor \lambda_l^\gamma \beta^{\gamma(l - h_{n_l,\varepsilon})} \rfloor$ and $K_l^\lambda := \lambda \beta^l$ then denote $\overline{F}_\infty(x) := \mathbf{P}(Z_\infty > x)$. Proposition 4.6.1 follows directly from Proposition 2.3.1.

Proposition 4.6.1. *Suppose $\gamma < 1$ and properties 1 and 2 hold then*

$$S^l_{M^\lambda_l}/K^\lambda_l \xrightarrow{d} R_{d_\lambda,0,\mathscr{L}_\lambda}$$

where $R_{d_\lambda,0,\mathscr{L}_\lambda}$ has an infinitely divisible law with drift

$$d_\lambda = \lambda^{1+\gamma}(1-\beta^{-\gamma})\sum_{K\in\mathbb{Z}}\beta^{(1+\gamma)K}\mathbf{E}\left[\frac{Z_\infty}{(\lambda\beta^K)^2+(Z_\infty)^2}\right],$$

0 variance and Lévy spectral function $\mathscr{L}_\lambda$ satisfying $\mathscr{L}_\lambda(x) = \lambda^\gamma\mathscr{L}_1(\lambda x)$ for all $\lambda > 0, x \in \mathbb{R}$ with $\mathscr{L}_1(x) = 0$ for $x < 0$ and

$$\mathscr{L}_1(x) = -(1-\beta^{-\gamma})\sum_{K\in\mathbb{Z}}\beta^{K\gamma}\overline{F}_\infty(x\beta^K)$$

for $x \geq 0$.

Combining this with the remark at the beginning of the section with $\lambda = (tC_\mathcal{D})^{1/\gamma} = (tc_\mu\mathbf{E}[\xi^*-1])^{1/\gamma}$ and that eventually $l = h_{n_l,0}$ we have that

$$\frac{\Delta_{n_l(t)}}{(C_\mathcal{D}n_l(t))^{\frac{1}{\gamma}}} \xrightarrow{d} R_{d_{(tC_\mathcal{D})^{1/\gamma}},0,\mathscr{L}_{(tC_\mathcal{D})^{1/\gamma}}}$$

which proves Theorem 4.2.

Proof of Theorem 4.3 (IVIE)

In IVIE, write $\gamma_\alpha := (\alpha-1)\log(\mu^{-1})/\log(\beta) = (\alpha-1)\gamma$. By (4.12) we have that

$$\mathbf{P}(\mathcal{H}(\mathcal{T}^{*-}) > n) \sim c_\mu^{\alpha-1}\Gamma(2-\alpha)\mathbf{P}(\xi^* \geq \mu^{-n}) \sim C_{\mu,\alpha}\beta^{-\gamma_\alpha n}L\left(\beta^{\gamma n}\right)$$

for a known constant $C_{\mu,\alpha}$. Due to the slowly varying term, we cannot apply Proposition 4.6.1 directly however Proposition 4.6.1 is proved using Proposition 2.3.1. It will therefore suffice to show convergence of the drift, variance and Lévy spectral function.

Recall that we consider subsequences $n_l(t)$ such that $a_{n_l(t)} \sim t\mu^{-l}$. From Propositions 4.5.10 and 4.5.14 we then have that for any $K \in \mathbb{Z}$ the laws of $\zeta_i^{l,K}$ converge to the laws of Z_∞ as $l \to \infty$. Let $(Z_\infty^{(i)})_{i\geq 1}$ be an independent sequence of variables with this law and denote $\overline{F}_\infty(x) := \mathbb{P}(Z_\infty > x)$. By Lemma 4.5.15 $\exists Z_{sup}$ such that $\zeta_i^{l,K} \preceq Z_{sup}$ for all $l \in \mathbb{N}, K \geq -(l-h_{n_l,\varepsilon})$ and $\mathbf{E}[Z_{sup}^{\gamma_\alpha+\epsilon}] < \infty$ for some $\epsilon > 0$; we denote $\overline{F}_{sup}(x) := \mathbf{P}(Z_{sup} > x)$. For $(\lambda_l)_{l\geq 0}$ converging to $\lambda > 0$ define $K_l^\lambda := \lambda\beta^l$ and for $\tilde{C}_{\mu,\alpha} = \mu^{-1}(2-\alpha)/(\alpha-1)$

$$M_l^\lambda := \left\lfloor \lambda_l^{\gamma_\alpha}\beta^{\gamma_\alpha l}\frac{\mathbf{P}(\xi^* > \mu^{-h_{n_l,\varepsilon}})}{\tilde{C}_{\mu,\alpha}L(\mu^{-h_{n_l,0}})}\right\rfloor.$$

Proposition 4.6.2. *In IVIE, for any $\lambda > 0$, as $l \to \infty$*

$$\sum_{i=1}^{M_l^\lambda} \frac{\tilde{\chi}_{n_l}^{i*}}{K_l^\lambda} \xrightarrow{d} R_{d_\lambda,0,\mathscr{L}_\lambda}$$

where

$$d_\lambda = \lambda^{1+\gamma_\alpha}(1-\beta^{-\gamma_\alpha}) \sum_{K\in\mathbb{Z}} \beta^{(1+\gamma_\alpha)K} \mathbb{E}\left[\frac{Z_\infty}{(\lambda\beta^K)^2 + (Z_\infty)^2}\right],$$

$$\mathscr{L}_\lambda(x) = \begin{cases} 0 & x \leq 0; \\ -\lambda^{\gamma_\alpha}(1-\beta^{-\gamma_\alpha}) \sum_{K\in\mathbb{Z}} \beta^{K\gamma_\alpha} \overline{F}_\infty(\lambda x \beta^{K\gamma_\alpha}) & x > 0. \end{cases}$$

Proof. By Proposition 2.3.1 it suffices to show the following:

1. for all $\epsilon > 0$
$$\lim_{l\to\infty} \mathbb{P}\left(\frac{\tilde{\chi}_{n_l}^{1*}}{K_l^\lambda} > \epsilon\right) = 0;$$

2. for all x continuity points
$$\mathscr{L}_\lambda(x) = \begin{cases} 0 & x \leq 0, \\ -\lim_{l\to\infty} M_l^\lambda \mathbb{P}\left(\frac{\tilde{\chi}_{n_l}^{1*}}{K_l^\lambda} > x\right) & x > 0; \end{cases}$$

3. for all $\tau > 0$ continuity points of $\mathscr{L}$
$$d_\lambda = \lim_{l\to\infty} M_l^\lambda \mathbb{E}\left[\frac{\tilde{\chi}_{n_l}^{1*}}{K_l^\lambda} \mathbf{1}_{\left\{\tilde{\chi}_{n_l}^{1*} \leq \tau K_l^\lambda\right\}}\right] + \int_{|x|\geq\tau} \frac{x}{1+x^2} \mathrm{d}\mathscr{L}_\lambda(x) - \int_{\tau\geq|x|>0} \frac{x^3}{1+x^2} \mathrm{d}\mathscr{L}_\lambda(x);$$

4. $$\lim_{\tau\to 0} \limsup_{l\to\infty} M_l^\lambda \mathrm{Var}_{\mathbb{P}}\left(\frac{\tilde{\chi}_{n_l}^{1*}}{K_l^\lambda} \mathbf{1}_{\left\{\tilde{\chi}_{n_l}^{1*} \leq \tau K_l^\lambda\right\}}\right) = 0.$$

We prove each of these in turn but we start by introducing a relation which will be fundamental to proving the final parts. For $K \in \mathbb{Z}$ let $c_l^K = \mathbf{P}(\mathcal{H}(\mathcal{T}^{*-}) > l + K | \mathcal{H}(\mathcal{T}^{*-}) > h_{n_l,\varepsilon})$ denote the probability that a deep branch is of height greater than $l+K$. Then by the asymptotic (4.12) we have that, for K such that $l+K \geq h_{n_l,\varepsilon}$, as $l \to \infty$

$$c_l^K = \frac{\mathbf{P}(\mathcal{H}(\mathcal{T}^{*-}) > l+K)}{\mathbf{P}(\mathcal{H}(\mathcal{T}^{*-}) > h_{n_l,\varepsilon})} \sim \mu^{(\alpha-1)K} \frac{\mathbf{P}(\mathcal{H}(\mathcal{T}^{*-}) > l)}{\mathbf{P}(\mathcal{H}(\mathcal{T}^{*-}) > h_{n_l,\varepsilon})}.$$

In particular, using (4.12) and that $\beta^{\gamma_\alpha} = \mu^{-(\alpha-1)}$

$$\begin{aligned} M_l^\lambda c_l^K &\sim \lambda^{\gamma_\alpha} \left(\frac{\mathbf{P}(\xi^* > \mu^{h_{n_l,\varepsilon}})}{\mathbf{P}(\mathcal{H}(\mathcal{T}^{*-}) > h_{n_l,\varepsilon})} \right) \left(\frac{\mathbf{P}(\mathcal{H}(\mathcal{T}^{*-}) > h_{n_l,0})}{\mathbf{P}(\xi^* > \mu^{h_{n_l,0}})} \right) \frac{\beta^{-\gamma_\alpha l}}{\mu^{-(\alpha-1)l}} \mu^{(\alpha-1)K} \\ &\sim \lambda^{\gamma_\alpha} \beta^{-\gamma_\alpha K} \end{aligned}$$

thus $M_l^\lambda (c_l^K - c_l^{K+1}) \to \lambda^{\gamma_\alpha} \beta^{-\gamma_\alpha K} (1 - \beta^{-\gamma_\alpha})$ and for any $\epsilon > 0$ and large enough l

$$M_l^\lambda c_l^K \leq C_\epsilon \lambda^{\gamma_\alpha} \beta^{-\gamma_\alpha K} \beta^{\frac{\epsilon}{2}|K|}. \tag{4.65}$$

To prove (1), notice that

$$\mathbb{P}\left(\frac{\tilde{X}_{n_l}^{1*}}{K_l^\lambda} > \epsilon \right) \leq \mathbf{P}\left(\mathcal{H}(\mathcal{T}^{*-}) \geq h_{n,\varepsilon/2} \middle| \mathcal{H}(\mathcal{T}^{*-}) \geq h_{n,\varepsilon} \right) + \mathbb{P}\left(\beta^{h_{n,\varepsilon/2} - l} Z_{sup} > \lambda \epsilon \right).$$

Both terms converge to 0 as $l \to \infty$ by the tail formula of a branch (4.12), the fact that Z_{sup} has no atom at ∞ and that $\beta^{h_{n_l}^{\varepsilon/2} - l} \to 0$ which follows from $l \sim h_{n,0}$.

For (2), recall that $\overline{F}_K^l(x) = \mathbb{P}(\zeta_i^{l,K} > x) = \mathbb{P}^K \left(\tilde{X}_{n_l}^{1*} \beta^{-(l+K)} > x \right)$, therefore

$$\begin{aligned} M_l^\lambda \mathbb{P}\left(\frac{\tilde{X}_{n_l}^{1*}}{K_l^\lambda} > x \right) &= \sum_{K \in \mathbb{Z}} \mathbf{1}_{\left\{K \geq -(l - h_{n_l,\varepsilon})\right\}} M_l^\lambda \mathbf{P}(\mathcal{H}(\mathcal{T}^{*-}) = l + K) \mathbb{P}^K \left(\frac{\tilde{X}_{n_l}^{1*}}{K_l^\lambda} > x \right) \\ &= \sum_{K \in \mathbb{Z}} \mathbf{1}_{\left\{K \geq -(l - h_{n_l,\varepsilon})\right\}} M_l^\lambda (c_l^K - c_l^{K+1}) \overline{F}_K^l (\lambda \beta^{-K} x). \end{aligned}$$

If $x > 0$ is a continuity point of $\mathscr{L}_\lambda$ then $\lambda x \beta^{-K}$ is a continuity point of $\overline{F}_\infty$ hence for any $K \in \mathbb{Z}$ as $l \to \infty$

$$\mathbf{1}_{\left\{K \geq -(l - h_{n_l,\varepsilon})\right\}} M_l^\lambda (c_l^K - c_l^{K+1}) \overline{F}_K^l (\lambda \beta^{-K} x) \to \lambda^{\gamma_\alpha} \beta^{-\gamma_\alpha K} (1 - \beta^{-\gamma_\alpha}) \overline{F}_\infty (\lambda \beta^{-K} x).$$

We need to exchange the sum and the limit; we do this using dominated convergence. Since $\gamma_\alpha < 1$ we can choose $\epsilon > 0$ such that $\gamma_\alpha + \epsilon < 1$ and $\epsilon < \gamma_\alpha$. By (4.65), for l sufficiently large $M_l^\lambda c_l^K \leq C_{\epsilon,\lambda} \beta^{-\gamma_\alpha K} \beta^{\frac{\epsilon}{2}|K|}$ hence

$$\sum_{K \geq -(l - h_{n_l,\varepsilon})} M_l^\lambda (c_l^K - c_l^{K+1}) \overline{F}_K^l (\lambda \beta^{-K} x) \leq C \sum_{K \in \mathbb{Z}} \overline{F}_{sup} (\lambda x \beta^{-K}) \beta^{-\gamma_\alpha K} \beta^{\frac{\epsilon}{2}|K|}.$$

Since Z_{sup} has moments up to $\gamma_\alpha + \epsilon$ we have that for $y = \lambda x$

$$\sum_{K < 0} \overline{F}_{sup} (\lambda x \beta^{-K}) \beta^{-\gamma_\alpha K} \beta^{\frac{\epsilon}{2}|K|} = \mathbb{E}\left[\sum_{K > 0} \mathbf{1}_{\{Z_{sup} > y \beta^K\}} \beta^{K(\gamma_\alpha + \epsilon/2)} \right]$$

$$
\begin{aligned}
&= \mathbb{E}\left[\sum_{K=1}^{\left\lfloor \frac{\log(Z_{sup}/y)}{\log(\beta)} \right\rfloor} \beta^{K(\gamma_\alpha+\epsilon/2)}\right] \\
&\leq C_y \mathbb{E}\left[Z_{sup}^{\gamma_\alpha+\frac{\epsilon}{2}}\right]
\end{aligned}
$$

which is finite. By choice of ϵ it follows that

$$
\sum_{K\geq 0} \overline{F}_{sup}(\lambda x \beta^{-K})\beta^{-\gamma_\alpha K}\beta^{\frac{\epsilon}{2}|K|} \leq \sum_{K\geq 0}\beta^{\left(\frac{\epsilon}{2}-\gamma_\alpha\right)K} < \infty.
$$

It therefore follows that for $x > 0$

$$
-\lim_{l\to\infty} M_l^\lambda \mathbb{P}\left(\frac{\tilde{\chi}_{n_l}^{1*}}{K_l^\lambda} > x\right) = -\lambda^{\gamma_\alpha}(1-\beta^{-\gamma_\alpha})\sum_{K\in\mathbb{Z}} \overline{F}_\infty(\lambda x\beta^{\gamma_\alpha K})\beta^{\gamma_\alpha K}.
$$

Moreover, for $x < 0$ we have that $\mathbb{P}\left(\tilde{\chi}_{n_l}^{1*}/K_l^\lambda < x\right) = 0$ which gives (2).

For (3) we have that $\int_0^\tau x \mathrm{d}\mathscr{L}_\lambda$ is well defined therefore

$$
\int_\tau^\infty \frac{x}{1+x^2}\mathrm{d}\mathscr{L}_\lambda - \int_0^\tau \frac{x^3}{1+x^2}\mathrm{d}\mathscr{L}_\lambda = \int_0^\infty \frac{x}{1+x^2}\mathrm{d}\mathscr{L}_\lambda - \int_0^\tau x\mathrm{d}\mathscr{L}_\lambda.
$$

We therefore want to show that

$$
\lim_{l\to\infty}\frac{M_l^\lambda}{K_l^\lambda}\mathbb{E}\left[\tilde{\chi}_{n_l}^{1*}\mathbf{1}_{\left\{\tilde{\chi}_{n_l}^{1*}\leq \tau K_l^\lambda\right\}}\right] = \int_0^\tau x\mathrm{d}\mathscr{L}_\lambda.
$$

Write $G_K^l(u) = \mathbb{E}\left[\zeta_1^{l,K}\mathbf{1}_{\{\zeta_1^{l,K}\leq u\}}\right]$ and $G_\infty(u) = \mathbb{E}[Z_\infty \mathbf{1}_{\{Z_\infty\leq u\}}]$. Then we have that

$$
\frac{M_l^\lambda}{K_l^\lambda}\mathbb{E}\left[\tilde{\chi}_{n_l}^{1*}\mathbf{1}_{\left\{\tilde{\chi}_{n_l}^{1*}\leq \tau K_l^\lambda\right\}}\right] = \lambda^{-1}\sum_{K\geq -(l-h_{n_l,\varepsilon})} M_l^\lambda(c_l^K - c_l^{K+1})\beta^K G_K^l(\tau\lambda\beta^{-K}).
$$

For each $K\in\mathbb{Z}$, as $l\to\infty$ we have that

$$
M_l^\lambda(c_l^K - c_l^{K+1})\beta^K G_K^l(\tau\lambda\beta^{-K}) \to \lambda^{\gamma_\alpha}(1-\beta^{-\gamma_\alpha})\beta^{(1-\gamma_\alpha)K}G_\infty(\tau\lambda\beta^{-K}).
$$

We want to exchange the limit and the sum which we do by dominated convergence. For any $\kappa\geq 0$ and random variable Y we have that $\mathbf{E}[Y\mathbf{1}_{\{Y\leq u\}}] \leq u^\kappa \mathbf{E}[Y^{1-\kappa}\mathbf{1}_{\{Y\leq u\}}]$. Using this with $u = \tau\lambda\beta^{-K}$ where $\kappa = 1-\gamma_\alpha - 2\epsilon/3$ for $K < 0$ and $\kappa = 1$ for $K \geq 0$, alongside (4.65) we have that

$$
\begin{aligned}
&\sum_{K\in\mathbb{Z}}\mathbf{1}_{\{K\geq -(l-h_{n_l,\varepsilon})\}} M_l^\lambda(c_l^K - c_l^{K+1})\beta^K G_K^l(\tau\lambda\beta^{-K}) \\
&\qquad \leq \sum_{K\geq 0} M_l^\lambda(c_l^K - c_l^{K+1})\beta^K\tau\lambda\beta^{-K}
\end{aligned}
$$

$$+\sum_{K<0} M_l^\lambda (c_l^K - c_l^{K+1})\beta^K \left(\beta^{\frac{2\epsilon}{3}K}(\tau\lambda)^{1-\gamma_\alpha-\frac{2\epsilon}{3}}\mathbf{E}\left[Z_{sup}^{\gamma_\alpha+\frac{2\epsilon}{3}}\right]\beta^{(\gamma_\alpha-1)K}\right)$$

$$\le C_\lambda \tau \sum_{K\ge 0}\beta^{-(\gamma_\alpha-\epsilon/2)K} + C_\lambda \tau^{1-\gamma_\alpha-\frac{2\epsilon}{3}}\mathbf{E}\left[Z_{sup}^{\gamma_\alpha+\frac{2\epsilon}{3}}\right]\sum_{K<0}\beta^{\frac{\epsilon}{6}K}$$

which is finite since $\gamma_\alpha > \epsilon/2$ and Z_{sup} has moments up to $\gamma_\alpha + \epsilon$. We therefore have that

$$\lim_{l\to\infty}\frac{M_l^\lambda}{K_l^\lambda}\mathbf{E}\left[\tilde{\chi}_{n_l}^{1*}\mathbf{1}_{\left\{\tilde{\chi}_{n_l}^{1*}\le\tau K_l^\lambda\right\}}\right] = \lambda^{\gamma_\alpha-1}(1-\beta^{-\gamma_\alpha})\sum_{K\in\mathbb{Z}}\beta^{K(\gamma_\alpha-1)}G_\infty(\tau\lambda\beta^K).$$

By definition we have that

$$\begin{aligned}\int_0^\tau x\mathrm{d}\mathscr{L}_\lambda &= \lambda^{\gamma_\alpha}(1-\beta^{-\gamma_\alpha})\int_0^\tau x\sum_{K\in\mathbb{Z}}\beta^{\gamma_\alpha K}\mathrm{d}(-\overline{F}_\infty)(\lambda x\beta^K)\\ &= \lambda^{\gamma_\alpha-1}(1-\beta^{-\gamma_\alpha})\sum_{K\in\mathbb{Z}}\beta^{(\gamma_\alpha-1)K}\int_{\lambda x\beta^K\le\lambda\tau\beta^K}\lambda x\beta^K\mathrm{d}(-\overline{F}_\infty)(\lambda x\beta^K)\\ &= \lambda^{\gamma_\alpha-1}(1-\beta^{-\gamma_\alpha})\sum_{K\in\mathbb{Z}}\beta^{(\gamma_\alpha-1)K}G_\infty(\tau\lambda\beta^K).\end{aligned}$$

It therefore remains to calculate $\int_0^\infty \frac{x}{1+x^2}\mathrm{d}\mathscr{L}_\lambda$.

$$\begin{aligned}\int_0^\infty \frac{x}{1+x^2}\mathrm{d}\mathscr{L}_\lambda &= \lambda^{\gamma_\alpha}(1-\beta^{-\gamma_\alpha})\int_0^\infty\frac{x}{1+x^2}\sum_{K\in\mathbb{Z}}\beta^{\gamma_\alpha K}\mathrm{d}(-\overline{F}_\infty)(\lambda x\beta^K)\\ &= \lambda^{\gamma_\alpha+1}(1-\beta^{-\gamma_\alpha})\sum_{K\in\mathbb{Z}}\beta^{(\gamma_\alpha+1)K}\mathbb{E}\left[\frac{Z_\infty}{(\lambda\beta^K)^2+(Z_\infty)^2}\right].\end{aligned}$$

The final sum is finite since for $K<0$

$$\beta^{(\gamma_\alpha+1)K}\mathbb{E}\left[\frac{Z_\infty}{(\lambda\beta^K)^2+(Z_\infty)^2}\right] = \lambda^{-1}\beta^{\gamma_\alpha K}\mathbb{E}\left[\frac{\lambda\beta^K Z_\infty}{(\lambda\beta^K)^2+(Z_\infty)^2}\right]\le\lambda^{-1}\beta^{\gamma_\alpha K}$$

which is summable. For $K\ge 0$

$$\begin{aligned}\mathbb{E}\left[\frac{Z_\infty}{(\lambda\beta^K)^2+(Z_\infty)^2}\right] &\le \mathbb{E}\left[\frac{Z_\infty}{(\lambda\beta^K)^2}\mathbf{1}_{\{Z_\infty\le\lambda\beta^K\}}+Z_\infty^{-1}\mathbf{1}_{\{Z_\infty^{-1}<(\lambda\beta^K)^{-1}\}}\right]\\ &\le \mathbb{E}\left[Z_\infty^{\gamma_\alpha+\epsilon/2}\right](\lambda\beta)^{-K(1+\gamma_\alpha+\epsilon/2)}\\ &\le C_\lambda\mathbb{E}\left[Z_{sup}^{\gamma_\alpha+\epsilon/2}\right]\beta^{-K(1+\gamma_\alpha+\epsilon/2)}\end{aligned}$$

which, multiplied by $\beta^{(\gamma_\alpha+1)K}$, is summable.

It now remains to prove (4). It suffices to show that

$$\lim_{\tau\to 0^+}\lim_{l\to\infty}\frac{M_l^\lambda}{(K_l^\lambda)^2}\mathbb{E}\left[(\tilde{\chi}_{n_l}^{1*})^2\mathbf{1}_{\left\{\tilde{\chi}_{n_l}^{1*}\le\tau K_l^\lambda\right\}}\right]=0. \tag{4.66}$$

Write $H^l_K(u) = \mathbb{E}\left[(\zeta_1^{l,K})^2\mathbf{1}_{\{\zeta_1^{l,K}\leq u\}}\right]$ then

$$\begin{aligned}\frac{M_l^\lambda}{(K_l^\lambda)^2}\mathbb{E}\left[(\tilde{\chi}_{n_l}^{1*})^2\mathbf{1}_{\left\{\tilde{\chi}_{n_l}^{1*}\leq\tau K_l^\lambda\right\}}\right] &= \frac{M_l^\lambda}{(K_l^\lambda)^2}\sum_{K\in\mathbb{Z}}(c_l^K - c_l^{K+1})\beta^{2(l+K)}H^l_K(\tau\lambda\beta^{-K}) \\ &\leq C_\lambda\sum_{K\in\mathbb{Z}}\beta^{(2-\gamma_\alpha)K}\beta^{\frac{\epsilon}{2}|K|}H^l_K(\tau\lambda\beta^{-K}).\end{aligned}$$

Using that, for any random variable Y, we have $\mathbb{E}[Y^2\mathbf{1}_{\{Y\leq u\}}] \leq u^\kappa\mathbb{E}[Y^{2-\kappa}\mathbf{1}_{\{Y\leq u\}}]$ with $u = \tau\lambda\beta^{-K}$ and $\kappa = 2$ it follows that

$$\sum_{K\geq 0}\beta^{(2-\gamma_\alpha)K}\beta^{\frac{\epsilon}{2}|K|}H^l_K(\tau\lambda\beta^{-K}) \;\leq\; C\tau^2\sum_{K\geq 0}\beta^{-(\gamma_\alpha-\epsilon/2)K} \;\leq\; C\tau^2$$

where the constant C depends on $\lambda, \beta, \gamma_\alpha$ and ϵ. Then, with $u = \tau\lambda\beta^{-K}$, $\kappa = 2-\gamma_\alpha - 2\epsilon/3$ we have that

$$H^l_K(\tau\lambda\beta^{-K})\beta^{(2-\gamma_\alpha)K} \leq \beta^{\frac{2\epsilon}{3}K}(\tau\lambda)^{2-\gamma_\alpha-\frac{2\epsilon}{3}}\mathbf{E}[Z_{sup}^{\gamma_\alpha+\frac{2\epsilon}{3}}]$$

and therefore

$$\sum_{K\leq 0}\beta^{(2-\gamma_\alpha)K}\beta^{\frac{\epsilon}{2}|K|}H^l_K(\tau\lambda\beta^{-K}) \;\leq\; C\tau^{2-\gamma_\alpha-\frac{2\epsilon}{3}}\mathbf{E}[Z_{sup}^{\gamma_\alpha+\frac{2\epsilon}{3}}]\sum_{K\leq 0}\beta^{\frac{\epsilon}{6}K} \;\leq\; C\tau^{2-\gamma_\alpha-\frac{2\epsilon}{3}}.$$

Since $\gamma_\alpha + \frac{2\epsilon}{3} < 1$ we have that (4.66) holds. □

Combining Proposition 4.6.2 with Corollary 4.3.4 and Lemma 4.3.5 with

$$\lambda = \Gamma(2-\alpha)^{\frac{1}{\gamma_\alpha}}c_\mu^{\frac{1}{\gamma}}\beta^{\frac{\log(t)}{\log(\mu^{-1})}-\left\lfloor\frac{\log(t)}{\log(\mu^{-1})}\right\rfloor}$$

proves Theorem 4.3.

4.7 Tightness of the random sum

We conclude the results for the walk on the subcritical tree in the sub-ballistic regime with Theorem 4.4 which is a tightness result for the process and a convergence result for the scaling exponent. We only prove the result in IVIE since the proof is standard (similar to [10, Theorem 1.1]) and the other cases follow by the same method. Recall that r_n is a_n in IVFE, $n^{1/\gamma}$ in FVIE, $a_n^{1/\gamma}$ in IVIE and $\overline{r}_n := \max\{m \geq 0 : r_m \leq n\}$.

Proof of Theorem 4.4 in IVIE. To prove statement 1 we will show that

$$\lim_{t\to\infty}\limsup_{n\to\infty}\mathbb{P}\left(\Delta_n/r_n \notin [t^{-1}, t]\right) = 0.$$

Let l be such that $a_{n_l(1)} \leq a_n < a_{n_{l+1}(1)}$ then by monotonicity of Δ_n

$$\mathbb{P}\left(\frac{\Delta_n}{a_n^{1/\gamma}} \notin [t^{-1},t]\right) \leq \mathbb{P}\left(\frac{\Delta_{n_l(1)}}{a_{n_{l+1}(1)}^{1/\gamma}} < t^{-1}\right) + \mathbb{P}\left(\frac{\Delta_{n_{l+1}(1)}}{a_{n_l(1)}^{1/\gamma}} > t\right).$$

Recall that R_t denotes the limiting distribution. The distribution of R_1 is continuous by [66, Theorem III.2] since $\lim_{x\to 0}\mathscr{L}(x) = -\infty$; therefore, since the sequence $(a_{n_{l+1}(1)}/a_{n_l(1)})^{1/\gamma}$ can be bounded above by some constant c,

$$\lim_{t\to\infty}\limsup_{n\to\infty}\mathbb{P}\left(\Delta_n/r_n \notin [t^{-1},t]\right) \leq \lim_{t\to\infty}\mathbb{P}\left(R_1 \notin [(tc)^{-1},tc]\right) = 0.$$

For statement 2 we want to show that

$$\lim_{t\to\infty}\limsup_{n\to\infty}\mathbb{P}\left(|X_n|/\overline{r}_n \notin [t^{-1},t]\right) = 0.$$

To do this we compare $|X_n|$ with Δ_n. In order to deal with the depth X_n reaches into the traps we use a bound for the height of a trap; for any $\epsilon > 0$ we have

$$\mathbb{P}\left(\frac{|X_n|}{\overline{r}_n} \geq t\right) \leq \mathbb{P}\left(\Delta_{\lfloor t\overline{r}_n - \overline{r}_n^\epsilon\rfloor} \leq n\right) + (t\overline{r}_n - \overline{r}_n^\epsilon)\mathbb{P}\left(\mathcal{H}(\mathcal{T}^{*-}) \geq \overline{r}_n^\epsilon\right).$$

By (4.12) we have that $(t\overline{r}_n - \overline{r}_n^\epsilon)\mathbb{P}\left(\mathcal{H}(\mathcal{T}^{*-}) \geq \overline{r}_n^\epsilon\right) \to 0$ as $n\to\infty$. Using the definition of $\overline{r}_n$ we have that

$$\mathbb{P}\left(\Delta_{\lfloor t\overline{r}_n - \overline{r}_n^\epsilon\rfloor} \leq n\right) \leq \mathbb{P}\left(\frac{\Delta_{\lfloor t\overline{r}_n - \overline{r}_n^\epsilon\rfloor}}{a_{t\overline{r}_n - \overline{r}_n^\epsilon}^{1/\gamma}} \leq \frac{a_{\overline{r}_n+1}^{1/\gamma}}{a_{t\overline{r}_n - \overline{r}_n^\epsilon}^{1/\gamma}}\right).$$

Since $a_{\overline{r}_n+1}^{1/\gamma}/a_{t\overline{r}_n - \overline{r}_n^\epsilon}^{1/\gamma}$ converges to t^{-1/γ_α} as $n\to\infty$, by continuity of the distribution of R_1 and statement 1 we have that $\lim_{t\to\infty}\limsup_{n\to\infty}\mathbb{P}\left(|X_n|/\overline{r}_n > t\right) = 0$.

It remains to show that $\lim_{t\to\infty}\limsup_{n\to\infty}\mathbb{P}\left(|X_n|/\overline{r}_n < t^{-1}\right) = 0$. We need to bound how far the walker backtracks after reaching a new furthest point in order to compare $|X_n|$ with Δ_n. Let $\zeta_0^Y := 0$ and for $j \geq 1$ define the j^{th} regeneration time of Y as $\zeta_j^Y := \min\{m > \zeta_{j-1}^Y : \{Y_n\}_{n=0}^{m-1} \cap \{Y_n\}_{n=m}^{\infty} = \emptyset\}$ and the j^{th} regeneration point as $\varrho_j := Y_{\zeta_j^Y}$ then

$$\max_{i<j\leq n}(|X_i| - |X_j|) \leq |\varrho_1| \vee \max_{2\leq i\leq n}(|\varrho_i| - |\varrho_{i-1}|) + \max_{0\leq i\leq n}\mathcal{H}(\mathcal{T}_{\rho_i}^{*-}).$$

The regeneration distances $(|\varrho_i| - |\varrho_{i-1}|)$, $|\varrho_1|$ and the heights of branches $\mathcal{H}(\mathcal{T}_{\rho_i}^{*-})$ have exponential moments for all i by comparison with a biased random walk on $\mathbb{Z}$

and Lemma 2.3.7. It then follows from a union bound that for any $\epsilon > 0$

$$\lim_{n\to\infty} \mathbb{P}\left(\max_{i<j\le n} (|X_i| - |X_j|) > \overline{r}_n^{\epsilon}\right) = 0.$$

We then have that

$$\begin{aligned}\mathbb{P}\left(|X_n|/\overline{r}_n < t^{-1}\right) &\le \mathbb{P}\left(\max_{i<j\le n} |X_i| - |X_j| > \overline{r}_n^{\epsilon}\right) + \mathbb{P}\left(\Delta_{\lfloor t^{-1}\overline{r}_n + \overline{r}_n^{\epsilon}\rfloor} > n\right) \\ &\le o(1) + \mathbb{P}\left(\frac{\Delta_{\lfloor 2t^{-1}\overline{r}_n\rfloor}}{a_{2t^{-1}\overline{r}_n}^{1/\gamma}} > \frac{a_{\overline{r}_n}^{1/\gamma}}{a_{2t^{-1}\overline{r}_n}^{1/\gamma}}\right).\end{aligned}$$

Then, since $a_{\overline{r}_n}^{1/\gamma}/a_{2t^{-1}\overline{r}_n}^{1/\gamma} \to (t/2)^{1/\gamma_\alpha}$ as $n \to \infty$, by continuity of the distribution of R_1 and statement 1 we indeed have that $\lim_{t\to\infty} \limsup_{n\to\infty} \mathbb{P}\left(|X_n|/\overline{r}_n < t^{-1}\right) = 0$.

For the final statement notice that

$$\mathbb{P}\left(\lim_{n\to\infty} \frac{\log|X_n|}{\log(n)} \ne \gamma(\alpha-1)\right) = \mathbb{P}\left(\lim_{n\to\infty} \frac{\log|X_n|}{\log(\overline{r}_n)} \cdot \frac{\log(\overline{r}_n)}{\log(n)} \ne \gamma(\alpha-1)\right)$$

and since $\overline{r}_n = n^{\gamma(\alpha-1)}\tilde{L}(n)$ for some slowly varying function $\tilde{L}$ we have that as $n \to \infty$ $\log(\overline{r}_n)/\log(n) \to \gamma(\alpha-1)$ thus it suffices to show that the following is equal to 0

$$\mathbb{P}\left(\lim_{n\to\infty} \frac{\log|X_n|}{\log(\overline{r}_n)} \ne 1\right) \le \mathbb{P}\left(\limsup_{n\to\infty} \frac{\log|X_n|}{\log(\overline{r}_n)} > 1\right) + \lim_{t\to\infty} \mathbb{P}\left(\liminf_{n\to\infty} \frac{|X_n|}{\overline{r}_n} \le t^{-1}\right).$$

By Fatou's lemma we can bound

$$\lim_{t\to\infty} \mathbb{P}\left(\liminf_{n\to\infty} \frac{|X_n|}{\overline{r}_n} \le t^{-1}\right) \le \lim_{t\to\infty} \liminf_{n\to\infty} \mathbb{P}\left(|X_n|/\overline{r}_n \le t^{-1}\right)$$

which is equal to 0 by tightness of $(|X_n|/\overline{r}_n)_{n\ge 0}$.

For the first term we have

$$\begin{aligned}\mathbb{P}\left(\limsup_{n\to\infty} \frac{\log|X_n|}{\log(\overline{r}_n)} > 1\right) &= \lim_{\varepsilon\to 0^+} \mathbb{P}\left(\limsup_{n\to\infty} \frac{\log|X_n|}{\log(\overline{r}_n)} \ge 1+\varepsilon\right) \\ &\le \lim_{\varepsilon\to 0^+} \mathbb{P}\left(\lim_{n\to\infty} \frac{\sup_{k\le n}|X_k|}{\overline{r}_n^{1+\varepsilon}} \ge 1\right).\end{aligned}$$

Writing $D'(n) := \left\{\max_{i=0,\dots,n} \mathcal{H}(\mathcal{T}_{\rho_i}^{*-}) \le 4\log(a_n)/\log(\mu^{-1})\right\}$, by (4.12) we have that $\mathbb{P}(D'(n)^c) = o(n^{-2})$ thus $\mathbf{P}(D'(n)^c\ i.o.) = 0$. On $D'(n)$

$$\sup_{k\le n} |X_k| \le |X_n| + |\kappa_{n+1}| - |\kappa_n| + \frac{4\log(a_n)}{\log(\mu^{-1})}$$

where κ_n is the last regeneration point before time n. Therefore, since (by Lemma

2.3.7) the difference between regeneration points $|\kappa_{n+1}|-|\kappa_n|$ has exponential moments we have that $\mathbb{P}(\lim_{n\to\infty}\sup_{j\leq n}(|\kappa_{j+1}|-|\kappa_j|)\geq \overline{r}_n)=0$; hence,

$$\begin{aligned}\mathbb{P}\left(\lim_{n\to\infty}\frac{\sup_{k\leq n}|X_n|}{\overline{r}_n^{1+\varepsilon}}\geq 1\right) &\leq \mathbb{P}\left(\liminf_{n\to\infty}\frac{|X_n|}{\overline{r}_n^{1+\varepsilon}}\geq 1-o(1)\right)\\ &\leq \lim_{t\to\infty}\liminf_{n\to\infty}\mathbb{P}\left(\frac{|X_n|}{\overline{r}_n}\geq t\right)\end{aligned}$$

where the second inequality follows by Fatou's lemma. The result follows by tightness of $(|X_n|/\overline{r}_n)_{n\geq 0}$. □

Theorem 4.1 follows from Theorem 4.4, Proposition 4.4.5 and Corollary 4.3.5 with $\lambda = t$ since $nq_n \sim n^{\varepsilon}$. More specifically, since $R_{d_t,0,\mathscr{L}_t}$ is the infinitely divisible law with characteristic exponent

$$\begin{aligned}id_1t+\int_0^\infty e^{itx}-1-\frac{itx}{1+x^2}\mathrm{d}\mathscr{L}_1(x) &= \int_0^\infty e^{itx}-1\mathrm{d}\mathscr{L}_1(x)\\ &= t^{-(\alpha-1)}\int_0^\infty e^{ix}-1\mathrm{d}\mathscr{L}_1(x)\end{aligned}$$

by a change of variables we have that the laws of the process $(\Delta_{nt}/a_n)_{t\geq 0}$ converge weakly as $n\to\infty$ under $\mathbb{P}$ on $D_{M_1}([0,\infty),\mathbb{R})$ to the law of the stable subordinator with characteristic function $\varphi(t)=e^{-C_\alpha t^{\alpha-1}}$ where $C_{\alpha,\beta,\mu}=-\int_0^\infty e^{ix}-1\mathrm{d}\mathscr{L}_1(x)$. A straightforward calculation then shows that the Laplace transform is of the form

$$\varphi_t(s) := \mathbb{E}[e^{-sR_{d_t,0,\mathscr{L}_t}}] = e^{-ts^{\alpha-1}C_{\alpha,\beta,\mu}}$$

where

$$C_{\alpha,\beta,\mu}=\frac{\pi(\alpha-1)}{\sin(\pi(\alpha-1))}\cdot\left(\frac{\beta(1-\beta\mu)}{2(\beta-1)}\right)^{\alpha-1}. \tag{4.67}$$

Recall that Theorem 4.1 only holds under the M_1 topology. Despite this, we do obtain J_1 convergence for the position of the walk in Corollary 4.7.1 which follows from *continuity of the inverse at strictly increasing functions* (Proposition 2.1.1.v).

Corollary 4.7.1. *For IVFE, the laws of the process*

$$\left(\frac{|X_{nt}|}{\overline{r}_n}\right)_{t\geq 0}$$

converge weakly as $n\to\infty$ under $\mathbb{P}$ on $D_{J_1}([0,\infty),\mathbb{R})$ to R_t^{-1}.

Chapter 5

Stable regimes for the randomly trapped random walk

In this chapter we consider the randomly trapped random walk in the sub-diffusive regime. Specifically, we prove annealed convergence to stable processes with index $\alpha \in (0,1) \cup (1,2)$.

In Section 5.1 we consider the sub-ballistic phase. Specifically, when the holding times have infinite mean, in Theorem 5.1 we give conditions under which the clock process S_t can be rescaled to converge to a stable subordinator. We then use this to show that the position of the walk converges, after suitable rescaling, to the inverse of a stable subordinator.

Theorem 5.1. *Suppose $\beta > 1$ and that $\exists a_n = n^{1/\alpha}L(n)$ for some $\alpha \in (0,1)$ and L slowly varying such that for any $k \in \mathbb{N}$, $\lambda > 0$ and some function $f \not\equiv 0$,*

$$-n \log\left(\mathbf{E}\left[E^\omega\left[\exp\left(-\frac{\lambda}{a_n}\eta_0\right)\right]^k\right]\right) \sim f(k)\lambda^\alpha \tag{5.1}$$

as $n \to \infty$. Then,

1. *S_{nt}/a_n converges in $\mathbb{P}$-distribution as $n \to \infty$ on $D_{M_1}([0,\infty),\mathbb{R})$ to an α-stable subordinator $\mathcal{S}_t$.*

2. *$X_{a_n t}/n$ converges in $\mathbb{P}$-distribution as $n \to \infty$ on $D_U([0,\infty),\mathbb{R})$ to the process $\frac{\beta-1}{\beta+1}\mathcal{S}_t^{-1}$.*

The condition (5.1) means that the result holds for many classes of distributions where the holding times belong to the domain of attraction of a stable law. This includes the Bouchaud trap model (however this is already known from [78]) as we will show in Section 5.3.

In Section 5.2 we consider the case where the holding times have finite mean but infinite variance. In this setting we are able to apply the speed result from Chapter

3 but the fluctuations are too large to obtain a central limit theorem. Here, we prove conditions under which, after suitable centring and rescaling, the time taken to reach the nt^{th} level of the backbone converges to a stable process. We then use results of [76] (detailed in Chapter 2) to show that the centred and rescaled walk converges in distribution to a stable process.

Recall that ν_β is the speed of the walk determined in Theorem 3.1 and $\Delta_m := \inf\{t > 0 : X_t = m\}$ is the first hitting time of m. The main result of Section 5.2 is Theorem 5.2.

Theorem 5.2. *Suppose $\beta > 1$ and that $\exists a_n = n^{1/\alpha}L(n)$ for some $\alpha \in (1, 2)$ and L slowly varying such that for any $k \in \mathbb{N}$, $\lambda > 0$ and a real valued function $f \not\equiv 0$,*

$$n \log\left(\mathbf{E}\left[E^\omega\left[\exp\left(-\frac{\lambda}{a_n}(\eta_0 - \mathbb{E}[\eta_0])\right)\right]^k\right]\right) \sim f(k)\lambda^\alpha \tag{5.2}$$

as $n \to \infty$. Then,

1. *$(\Delta_{nt} - nt\nu_\beta^{-1})/a_n$ converges in $\mathbb{P}$-distribution as $n \to \infty$ on $D_{M_1}([0, \infty), \mathbb{R})$ to an α-stable process $\mathcal{V}_t^\alpha$.*
2. *$(X_{nt} - nt\nu_\beta)/a_n$ converges in $\mathbb{P}$-distribution as $n \to \infty$ on $D_{M_1}([0, \infty), \mathbb{R})$ to the α-stable process $-\nu_\beta \mathcal{V}_{\nu_\beta t}^\alpha$.*

The process $\mathcal{V}_t^\alpha$ can be decomposed into a positive jump process with negative drift. To see this notice that the random variables $\eta_k - \mathbb{E}[\eta_0]$ are bounded below and therefore have a one-sided heavy tail. It follows that $\Delta_{nt} - nt\nu_\beta$ is a totally asymmetric jump process. We also give an expression for the Laplace exponent of $\mathcal{V}_t^\alpha$ in (5.9) following the proof.

The two asymptotic conditions (5.1) and (5.2) for the sub-diffusive regimes appear to be quite technical however they relate to the usual stable law conditions. Specifically, η_0 belongs to the domain of attraction of a stable law of index α with respect to $\mathbb{P}$ if and only if the relevant asymptotic holds for $k = 1$ (e.g. [76, Chapter 4]). We need to extend to $k \geq 2$ to account for the walk visiting vertices multiple times.

These theorems cannot be applied directly to the random walk on the subcritical GW-tree conditioned to survive because of the lattice effect which we studied in greater detail in Chapter 4. That is, the asymptotics (5.1) and (5.2) will only hold along subsequences if at all. In Section 5.3 we will show that these asymptotics do indeed hold along subsequences for the comb model which can be viewed as a biased random walk on a subcritical GW-tree conditioned to survive which has been pruned so that it consists of the backbone and the unique self avoiding paths from the root of each branch to the deepest point. We expect that only the depth, and not the foliage, of each branch in the tree is important for the scaling therefore the comb model is an appropriate mechanism for the study of random walks on subcritical GW-trees.

It is worth noting that three of the four limits hold with the M_1 topology; it is natural to consider whether they can be extended to the J_1 topology. The main issue we encounter is that if the slowing is caused by spending a large amount of time in a few large traps in the environment then the total time spent in one of these traps may consist of several long excursions. This results in several large jumps in the discrete process S_{nt} contributing to a single large jump in the limit. This means that the convergence will not hold under the J_1 topology which distinguishes between a series of small jumps in quick succession and a single large jump of the combined magnitude. A similar argument can be used for the position of the walk and in Section 5.3 we use the example of random walk in random scenery to show that, in general, $(X_{nt} - nt\nu_\beta)/a_n$ does not converge in $D_{J_1}([0,\infty),\mathbb{R})$.

5.1 Sub-ballistic stable limits for the randomly trapped random walk

In this section we classify limits of the RTRW model where the embedded random walk has a positive bias and the holding times have infinite mean. The main aim is to prove Theorem 5.1 which we do by considering the arguments of [27] and applying the results of [76] outlined in Chapter 2. For $x \in \mathbb{Z}$ write

$$\hat{\omega}_x(\lambda) = \int e^{-\lambda u}\omega_x(\mathrm{d}u) = E^\omega[e^{-\lambda \eta_{x,1}}]$$

to be the quenched Laplace transform of ω_x. For convenience we write $\mathbb{P}^Y(\cdot) := \mathbb{P}(\cdot|Y)$ and $P^{\omega,Y}(\cdot) := P^\omega(\cdot|Y)$. We begin by showing that the one-dimensional distributions of the scaled clock process converge in Proposition 5.1.1.

Proposition 5.1.1. *Under the assumptions of Theorem 5.1, for some $c \in (0,\infty)$ and any $\lambda > 0$,*

$$\lim_{n\to\infty} \mathbb{E}\left[\exp\left(-\lambda\frac{S_{nt}}{a_n}\right)\right] = e^{-ct\lambda^\alpha}.$$

Proof. Conditional on Y, the holding times at different vertices are independent therefore

$$\begin{aligned}\mathbb{E}^Y\left[\exp\left(-\lambda\frac{S_{nt}}{a_n}\right)\right] &= \mathbb{E}^Y\left[\exp\left(-\frac{\lambda}{a_n}\sum_{x\in\mathbb{Z}}\sum_{i=1}^{\mathcal{L}(x,\lfloor nt\rfloor-1)}\eta_{x,i}\right)\right] \\ &= \prod_{x\in\mathbb{Z}}\mathbb{E}^Y\left[\exp\left(-\frac{\lambda}{a_n}\sum_{i=1}^{\mathcal{L}(x,\lfloor nt\rfloor-1)}\eta_{x,i}\right)\right].\end{aligned}$$

Under $P^{\omega,Y}$ the holding times at a given vertex are independent, identically distributed and do not depend on Y therefore we can write the above expression as

$$\prod_{x\in\mathbb{Z}}\mathbf{E}^Y\left[E^{\omega,Y}\left[\prod_{i=1}^{\mathcal{L}(x,\lfloor nt\rfloor-1)}e^{-\frac{\lambda}{a_n}\eta_{x,i}}\right]\right]=\prod_{x\in\mathbb{Z}}\mathbf{E}^Y\left[\hat{\omega}_x\left(\frac{\lambda}{a_n}\right)^{\mathcal{L}(x,\lfloor nt\rfloor-1)}\right]$$
$$=\prod_{k=1}^{\infty}\prod_{x\in\mathcal{R}_k(nt)}\mathbf{E}\left[\hat{\omega}_x\left(\frac{\lambda}{a_n}\right)^k\right]$$

where $\mathcal{R}_k(m)=\{x\in\mathbb{Z}:\ \mathcal{L}(x,m-1)=k\}$ is the collection of vertices with local time k at time $m-1$. By translation invariance we then have that

$$\begin{aligned}\mathbb{E}^Y\left[\exp\left(-\lambda\frac{S_{nt}}{a_n}\right)\right] &= \prod_{k=1}^{\infty}\mathbf{E}\left[\hat{\omega}_0\left(\frac{\lambda}{a_n}\right)^k\right]^{|\mathcal{R}_k(nt)|}\\ &= \exp\left(\sum_{k=1}^{\infty}|\mathcal{R}_k(nt)|\log\left(\mathbf{E}\left[\hat{\omega}_0\left(\frac{\lambda}{a_n}\right)^k\right]\right)\right).\end{aligned}$$

Write $\mathcal{R}_k^+(m)=\{x\in\mathbb{Z}:\ \mathcal{L}(x,m-1)\geq k\}$ then by [67] we have that, for fixed $t>0$, $|\mathcal{R}_k(nt)|/\lfloor nt\rfloor$ converges almost surely to $u_\infty^2(1-u_\infty)^{k-1}$ and $|\mathcal{R}_k^+(nt)|/\lfloor nt\rfloor$ converges almost surely to $u_\infty(1-u_\infty)^{k-1}$ as $n\to\infty$ where $u_\infty=(\beta-1)/(\beta+1)$ is the probability that the walk never returns to the origin. The local time at the origin stochastically dominates that of any other vertex therefore

$$\frac{E[|\mathcal{R}_k^+(m)|]}{m} = \frac{1}{m}\sum_{x=-m}^{m}P(\mathcal{L}(x,m)\geq k) \leq \frac{2m+1}{m}P(\mathcal{L}(0,m)\geq k).$$

The local time $\mathcal{L}(0,m)$ is increasing in m, therefore for each $k\geq 1$ we have that $P(\mathcal{L}(0,m)\geq k)\leq P(\mathcal{L}(0,\infty)\geq k)=(1-u_\infty)^{k-1}$ since $\mathcal{L}(0,\infty)$ is geometrically distributed with termination probability u_∞. It follows that

$$\frac{E[|\mathcal{R}_k^+(nt)|]}{nt}\leq C(1-u_\infty)^k \tag{5.3}$$

uniformly over n,t,k. For fixed $M\in\mathbb{N}$, by using the convergence of $|\mathcal{R}_k(nt)|/\lfloor nt\rfloor$ and assumption (5.1) we have that for P-a.e. Y

$$\sum_{k=1}^{M}|\mathcal{R}_k(nt)|\log\left(\mathbf{E}\left[\hat{\omega}_0\left(\frac{\lambda}{a_n}\right)^k\right]\right)\to -u_\infty^2\lambda^\alpha t\sum_{k=1}^{M}f(k)(1-u_\infty)^{k-1}.$$

By Jensen's inequality and (5.1),

$$-\log(\mathbf{E}[\hat{\omega}_0(\lambda/a_n)^k]) \leq -k\log(\mathbf{E}[\hat{\omega}_0(\lambda/a_n)]) \leq Ck\lambda^\alpha n^{-1}$$

thus f grows at most linearly in k and therefore we have that $\sum_{k=1}^{M} f(k)(1-u_\infty)^{k-1}$ converges as $M \to \infty$. Moreover,

$$-\sum_{k=M}^{\infty} |\mathcal{R}_k(nt)| \log\left(\mathbf{E}\left[\hat{\omega}_0\left(\frac{\lambda}{a_n}\right)^k\right]\right) \;\leq\; \frac{C\lambda^\alpha}{n}\sum_{k=M}^{\infty} k|\mathcal{R}_k(nt)|.$$

By Markov's inequality and (5.3) we have that for $c_1 \in (0, -\log(1-u_\infty))$ and some $\tilde{c} > 0$

$$P\left(\frac{|\mathcal{R}_k^+(nt)|}{nt} \geq e^{-c_1 k}\right) \;\leq\; E\left[\frac{|\mathcal{R}_k^+(nt)|}{nt}\right] e^{c_1 k} \;\leq\; C(1-u_\infty)^k e^{c_1 k} \;\leq\; Ce^{-\tilde{c}k}.$$

A union bound then shows that

$$P\left(\bigcup_{k\geq M}\left\{\frac{|\mathcal{R}_k(nt)|}{nt} \geq e^{-c_1 k}\right\}\right) \leq \sum_{k\geq M} e^{-\tilde{c}k} \leq Ce^{-\tilde{c}M}.$$

It then follows that, for $\delta > 0$,

$$P\left(-\sum_{k=M}^{\infty} |\mathcal{R}_k(nt)| \log\left(\mathbf{E}\left[\hat{\omega}_0\left(\frac{\lambda}{a_n}\right)^k\right]\right) \geq \delta\right) \;\leq\; Ce^{-\tilde{c}M} + \mathbf{1}_{\left\{Ct\lambda^\alpha \sum_{k=M}^{\infty} ke^{-\tilde{c}k} \geq \delta\right\}}$$

which converges to 0 as $M \to \infty$ independently of n. In particular,

$$\sum_{k=1}^{\infty} |\mathcal{R}_k(nt)| \log\left(\mathbf{E}\left[\hat{\omega}_0\left(\frac{\lambda}{a_n}\right)^k\right]\right) \to -u_\infty^2\lambda^\alpha t\sum_{k=1}^{\infty} f(k)(1-u_\infty)^{k-1}$$

in P-probability therefore, by bounded convergence, we have that

$$\lim_{n\to\infty} \mathbb{E}\left[\exp\left(-\lambda\frac{S_{nt}}{a_n}\right)\right] = e^{-u_\infty^2\lambda^\alpha t\sum_{k=1}^{\infty} f(k)(1-u_\infty)^{k-1}} = e^{-ct\lambda^\alpha}.$$

$\square$

Before proving the main result we prove a technical lemma which allows us to compare the process with a version in which we re-sample the environment at fixed times. This follows similarly to [27, Lemma 3.2] however we include the proof for brevity.

Fix $m \in \mathbb{N}$, $0 < t_1 < ... < t_m = t$ and let $\overline{\omega} := (\omega_x^j)_{x\in\mathbb{Z}, j=1,\dots,m}$ be a sequence of i.i.d. π random measures. For $x \in \mathbb{Z}$, $j = 1, ..., m$ and $i \geq 1$ let $\eta_{x,i}^j$ be independent with $\eta_{x,i}^j \sim \omega_x^j$. Let $j(x)$ be such that $nt_{j(x)-1} \leq \Delta_x^Y < nt_{j(x)}$ denote the index of the interval in which x is first reached. Define

$$S'_k := \sum_{x\in\mathbb{Z}} \sum_{i=1}^{\mathcal{L}(x,k-1)} \eta_{x,i}^{j(x)}$$

then $S' \overset{\mathrm{d}}{=} S$. The sum S'_k can be thought of as the sum of the first k holding times where we refresh the entire unseen environment at times nt_j. We then define the approximation of S':

$$S''_k = \sum_{j=1}^{m} \sum_{l=nt_{j-1}}^{(k\wedge nt_j)-1} \eta^j_{Y_l,\mathcal{L}(Y_l,l)}.$$

The sum S'' can be thought of as the sum of the first k holding times where we refresh the entire environment at times nt_j. By independence of the environment and the walk between times nt_{j-1} and nt_j we have that the differences $S''_{nt_j} - S''_{nt_{j-1}}$ are independent.

Lemma 5.1.2. *Under the assumptions of Theorem 5.1 we have that for any $t, \varepsilon > 0$,*

$$\mathbb{P}\left(\sup_{s\le t} |S''_{ns} - S'_{ns}| > \varepsilon a_n\right)$$

converges to 0 as $n \to \infty$.

Proof. Due to the coupling between S' and S'' we have that

$$\sup_{s\le t} |S''_{ns} - S'_{ns}| \le \sum_{j=1}^{m} \sum_{x\in M_j^n} \sum_{i=1}^{\mathcal{L}(x,\infty)} \eta^j_{x,i}$$

where $M_j^n := \{Y_k\}_{k=0}^{nt_j-1} \cap \{Y_k\}_{k=nt_j}^{\infty}$ is the set of vertices visited both before and after nt_j.

On the event that the walk never backtracks distance $C\log(n)$ we have that M_j^n contains at most $C\log(n)+1$ vertices. By a union bound we therefore have that

$$\begin{aligned}
&\mathbb{P}\left(\sup_{s\le t} |S''_{ns} - S'_{ns}| > \varepsilon a_n\right) \\
&\quad \le \tilde{C} m \log(n) \mathbb{P}\left(\sum_{i=1}^{\mathcal{L}(0,\infty)} \eta^1_{0,i} > \frac{C_{\varepsilon,m} a_n}{\log(n)}\right) + \mathbb{P}\left(\bigcup_{x\le nt} \bigcup_{l\ge \Delta_x^Y} \{x - Y_l > C\log(n)\}\right).
\end{aligned}$$

By Lemma 2.3.2, the probability that the walk started from x reaches the vertex $x-1$ is β^{-1} therefore we have that

$$\mathbb{P}\left(\bigcup_{x\leq nt}\bigcup_{l\geq \Delta_x^Y}\{x-Y_l>C\log(n)\}\right) \;\leq\; C_t n\beta^{-C\log(n)} \;=\; C_t n^{1-C\log(\beta)}$$

which converges to 0 as $n\to\infty$ for C sufficiently large.

The number of visits to the origin is geometrically distributed with return probability u_∞. We therefore have that $\mathbb{P}(\mathcal{L}(0,\infty)>\log(n))\leq n^{\log(u_\infty)}$; in particular,

$$\begin{aligned}\tilde{C}m\log(n)\mathbb{P}&\left(\sum_{i=1}^{\mathcal{L}(0,\infty)}\eta_{0,i}^1>\frac{C_{\varepsilon,m}a_n}{\log(n)}\right)\\ &\leq \tilde{C}m\log(n)^2\mathbb{P}\left(\eta_0>\frac{C_{\varepsilon,m}a_n}{\log(n)^2}\right)+\tilde{C}m\log(n)\mathbb{P}(\mathcal{L}(0,\infty)>\log(n))\end{aligned}$$

which converges to 0 as $n\to\infty$ by the assumptions of Theorem 5.1. $\square$

Using these two results we can conclude that the walk X, suitably scaled, converges in distribution to the inverse of a stable subordinator by applying results of [76].

Proof of Theorem 5.1. We begin by completing the result for the clock process S_t. By Proposition 5.1.1 we have that the one-dimensional distributions of S_{nt}/a_n converge and the limit $\mathcal{S}_t$ is $\mathbb{P}$-a.s. finite for any $t>0$. Moreover, $\mathbb{P}$-a.s. we have that $\mathcal{S}_0=0$ since $\lim_{t\to 0}e^{-ct\lambda^\alpha}=1$. It then follows from *equivalent characterisations of convergence for monotone functions* (Proposition 2.1.1.iii) and that S_{nt} in increasing in t that S_{nt}/a_n converges in distribution to the process $\mathcal{S}_t$ with respect to the M_1 topology. It remains to show that $\mathcal{S}$ has stationary, independent increments and is self-similar with index $1/\alpha$.

Stationary increments of $\mathcal{S}$ follows from stationary increments of S_m which holds because the traps in the environment are i.i.d. Independence of the increments follows from Lemma 5.1.2. To see that $\mathcal{S}$ is self-similar let $\lambda,t>0$ then,

$$\mathcal{S}_t \;=\; \lim_{n\to\infty}\frac{S(nt)}{a_n} \;=\; \lim_{n\to\infty}\frac{a_{\lambda n}}{a_n}\cdot\frac{S(\lambda nt/\lambda)}{a_{\lambda n}} \;\overset{\mathrm{d}}{=}\; \mathcal{S}_{t/\lambda}\lim_{n\to\infty}\frac{a_{\lambda n}}{a_n} \;=\; \lambda^{1/\alpha}\mathcal{S}_{t/\lambda}$$

so we indeed have that $\mathcal{S}$ is self-similar with index $1/\alpha$. Since $\mathcal{S}_t$ is increasing it therefore follows that $\mathcal{S}$ is an α-stable subordinator.

The limit process $\mathcal{S}_t$ belongs to $D([0,\infty),\mathbb{R})$, is unbounded and strictly increasing. By *continuity of the inverse at strictly increasing functions* (Proposition 2.1.1.v), the inverse map between unbounded, increasing functions in $D_{M_1}([0,\infty),\mathbb{R})$ onto $D_U([0,\infty),\mathbb{R})$ is continuous at unbounded, strictly increasing functions. Therefore, the sequence $S^{-1}_{a_nt}/n$ converges in $\mathbb{P}$-distribution on $D_U([0,\infty),\mathbb{R})$ to $\mathcal{S}_t^{-1}$.

The result then follows from *continuity of composition at continuous limits* (Proposition 2.1.1.i) since Y_{nt}/n converges $\mathbb{P}$-a.s. on $D_U([0,\infty),\mathbb{R})$ to the process $t(\beta-1)/(\beta+1)$ and $X_{a_nt}n^{-1} = Y_{S^{-1}_{a_nt}}n^{-1}$. $\square$

5.2 Sub-diffusive stable limits for the randomly trapped random walk

In this section we consider the regime where $\mathbb{E}[\eta_0] < \infty$ but $\mathbb{E}[\eta_0^2] = \infty$. In this setting we do not have a central limit theorem for X; however, by Theorem 3.1 we have that if $\beta > 1$ and $\mathbb{E}[\eta_0] < \infty$ then $X_{nt}n^{-1}$ converges $\mathbb{P}$-a.s. to $\nu_\beta t$ where

$$\nu_\beta = \frac{u_\infty}{\mathbb{E}[\eta_0]} \qquad \text{and} \qquad u_\infty = \frac{\beta-1}{\beta+1}.$$

The main aim of the section is to prove Theorem 5.2 which shows that X can be centred and rescaled so that it converges to a stable process.

Similarly to Chapter 4, we will consider the first hitting times $\Delta_k := \inf\{t \geq 0 : X_t = k\}$ and the analogue for the embedded walk $\Delta_k^Y := \inf\{m \geq 0 : Y_m = k\}$ in order to study the process X. We begin by exploiting the renewal structure of the walk. Let $\zeta_0^Y = 0$ and, for $j = 1, 2, ...,$ define $\zeta_j^Y := \inf\{m > \zeta_{j-1}^Y : \{Y_l\}_{l=0}^{m-1} \cap \{Y_l\}_{l=m}^{\infty} = \emptyset\}$ to be the regeneration times of the walk Y. We then have that $\zeta_j^X := S_{\zeta_j^Y}$ for $j \geq 1$ are regeneration times for X, $\varrho_j := X_{\zeta_j^X} = Y_{\zeta_j^Y}$ are regeneration points and we write

$$\chi_j \; := \; \left(X_{\zeta_j^X} - X_{\zeta_{j-1}^X} - \left(\zeta_j^X - \zeta_{j-1}^X\right)\nu_\beta\right) \; = \; \left(\varrho_j - \varrho_{j-1} - \left(\zeta_j^X - \zeta_{j-1}^X\right)\nu_\beta\right).$$

By Lemma 3.1.3 we have that $\{\chi_j\}_{j\geq 2}$ are centred and i.i.d. under $\mathbb{P}$ whenever $\beta > 1$ and $\mathbb{E}[\eta_0] < \infty$. For $m \in \mathbb{N}$ define

$$\Sigma_m := \sum_{j=2}^{m} \chi_j = \left(X_{\zeta_m^X} - \zeta_m^X \nu_\beta\right) - \left(X_{\zeta_1^X} - \zeta_1^X \nu_\beta\right)$$

then since χ_j are i.i.d. we can consider χ_2 and write

$$\chi_2 \; = \; \left(\varrho_2 - \varrho_1 - u_\infty(\zeta_2^Y - \zeta_1^Y)\right) + \nu_\beta\left(\mathbb{E}[\eta_0](\zeta_2^Y - \zeta_1^Y) - (\zeta_2^X - \zeta_1^X)\right).$$

Lemma 5.2.1. *Suppose $\beta > 1$ and $T < \infty$. If $a_n >> n^{1/2}$ then*

$$\sup_{t \leq T} \left| \sum_{j=2}^{nt} \frac{\varrho_j - \varrho_{j-1} - u_\infty(\zeta_j^Y - \zeta_{j-1}^Y)}{a_n} \right|$$

converges in $\mathbb{P}$ probability to 0 as $n \to \infty$.

Proof. Define

$$\psi_m := \sum_{j=2}^{m} \left(\varrho_j - \varrho_{j-1} - u_\infty(\zeta_j^Y - \zeta_{j-1}^Y)\right)$$

which is a martingale with centred and independent jumps. By Doob's inequality we then have that for $\varepsilon > 0$,

$$\mathbb{P}\left(\sup_{m\leq nT} |\psi_m| \geq \varepsilon a_n\right) \;\leq\; \frac{\mathbb{E}[\psi_{nT}^2]}{\varepsilon^2 a_n^2} \;\leq\; \frac{C_{T,\varepsilon} n}{a_n^2}\mathbb{E}\left[\left(\varrho_2 - \varrho_1 - u_\infty(\zeta_2^Y - \zeta_1^Y)\right)^2\right]$$

which converges to 0 as $n \to \infty$ since $n/a_n^2 \to 0$ and the time and distance between regenerations have exponential moments by Lemma 2.3.7. $\square$

We now prove a technical lemma that will allow us to use dominated convergence in the proof of Proposition 5.2.3 where we show that the random variables χ_j belong to the domain of attraction of a stable law.

Lemma 5.2.2. *Under the assumptions of Theorem 5.2 there exist constants c, C depending only on λ such that for sufficiently large n (independently of k)*

$$ck \;\leq\; n\log\left(\mathbf{E}\left[E^\omega\left[\exp\left(-\frac{\lambda}{a_n}(\eta_0 - \mathbb{E}[\eta_0])\right)\right]^k\right]\right) \;\leq\; Ck^2.$$

Proof. By Jensen's inequality, for a positive random variable Z and $k \in \mathbb{Z}$ satisfying $\mathbb{E}[Z^k] < \infty$ we have that

$$\mathbb{E}[Z]^k \;\leq\; \mathbf{E}\left[E^\omega\left[Z\right]^k\right] \;\leq\; \mathbb{E}[Z^k]. \tag{5.4}$$

It therefore follows that

$$k\log\left(\mathbf{E}\left[E^\omega\left[Z\right]\right]\right) \;\leq\; \log\left(\mathbf{E}\left[E^\omega\left[Z\right]^k\right]\right) \;\leq\; \log\left(\mathbf{E}\left[E^\omega\left[Z^k\right]\right]\right). \tag{5.5}$$

Choose

$$Z(\lambda) = \exp\left(-\frac{\lambda}{a_n}(\eta_0 - \mathbb{E}[\eta_0])\right) \quad \text{then} \quad Z(\lambda)^k = \exp\left(-\frac{k\lambda}{a_n}(\eta_0 - \mathbb{E}[\eta_0])\right) = Z(k\lambda)$$

however, by (5.2),

$$n\log\left(\mathbf{E}\left[E^\omega\left[Z\right]^k\right]\right) \sim f(k)\lambda^\alpha \qquad \text{and} \qquad n\log\left(\mathbf{E}\left[E^\omega\left[Z^k\right]\right]\right) \sim f(1)(k\lambda)^\alpha.$$

The bound (5.5) then implies that $f(1)k \leq f(k) \leq f(1)k^\alpha$. If $f(1) = 0$ then $f(k) = 0$ for all k however, since $f \not\equiv 0$, this cannot be the case. In particular, since $\alpha > 1$ we have that $f(1)k \leq f(1)k^\alpha$ which implies that $f(1) > 0$ and therefore $f(k) > 0$ for all k.

The lower bound in the statement of the theorem follows from (5.2) with $k=1$:

$$\begin{aligned}
n\log&\left(\mathbf{E}\left[E^{\omega}\left[\exp\left(-\frac{\lambda}{a_n}(\eta_0-\mathbb{E}[\eta_0])\right)\right]^k\right]\right)\\
&\geq kn\log\left(\mathbf{E}\left[E^{\omega}\left[\exp\left(-\frac{\lambda}{a_n}(\eta_0-\mathbb{E}[\eta_0])\right)\right]\right]\right)\\
&\sim kf(1)\lambda^{\alpha}.
\end{aligned}$$

For the upper bound, using (5.4) we have that

$$\begin{aligned}
n\log&\left(\mathbf{E}\left[E^{\omega}\left[\exp\left(-\frac{\lambda}{a_n}(\eta_0-\mathbb{E}[\eta_0])\right)\right]^k\right]\right)\\
&\leq n\log\left(\mathbb{E}\left[\exp\left(-\frac{\lambda k}{a_n}(\eta_0-\mathbb{E}[\eta_0])\right)\right]\right)\\
&=\frac{n\lambda k\mathbb{E}[\eta_0]}{a_n}+n\log\left(\mathbb{E}\left[\exp\left(-\frac{\lambda k}{a_n}\eta_0\right)\right]\right). \qquad (5.6)
\end{aligned}$$

Using integration by parts we then have that

$$\begin{aligned}
\mathbb{E}\left[\exp\left(-\frac{\lambda k}{a_n}\eta_0\right)\right] &= \int_0^\infty e^{-\lambda kx}\mathbb{P}\left(\frac{\eta_0}{a_n}\in \mathrm{d}x\right)\\
&= -e^{-k\lambda x}\mathbb{P}\left(\eta_0\geq xa_n\right)\Big|_0^\infty-\lambda k\int_0^\infty e^{-\lambda kx}\mathbb{P}\left(\eta_0\geq xa_n\right)\mathrm{d}x\\
&= 1-\lambda k\int_0^\infty e^{-\lambda kx}\mathbb{P}\left(\eta_0\geq xa_n\right)\mathrm{d}x.
\end{aligned}$$

Using that $1-e^{-y}\leq y$ for $y\geq 0$ we can then write

$$\begin{aligned}
-\lambda k&\int_0^\infty e^{-\lambda kx}\mathbb{P}\left(\eta_0\geq xa_n\right)\mathrm{d}x\\
&=-\lambda k\int_0^\infty \mathbb{P}\left(\eta_0\geq xa_n\right)\mathrm{d}x+\lambda k\int_0^\infty(1-e^{-\lambda kx})\mathbb{P}\left(\eta_0\geq xa_n\right)\mathrm{d}x\\
&\leq -\frac{\lambda k\mathbb{E}[\eta_0]}{a_n}+\lambda k\left(\int_0^1\lambda kx\mathbb{P}\left(\eta_0\geq xa_n\right)\mathrm{d}x+\int_1^\infty\mathbb{P}\left(\eta_0\geq xa_n\right)\mathrm{d}x\right).
\end{aligned}$$

Using integration by parts again we have that

$$\begin{aligned}
\int_0^1\lambda kx\mathbb{P}\left(\eta_0\geq xa_n\right)\mathrm{d}x &= \lambda k\left(\frac{x^2}{2}\mathbb{P}\left(\eta_0\geq xa_n\right)\Big|_0^1+\int_0^1\frac{x^2}{2}\mathbb{P}\left(\frac{\eta_0}{a_n}\in\mathrm{d}x\right)\right)\\
&=\frac{\lambda k}{2}\left(\mathbb{P}\left(\eta_0\geq a_n\right)+\mathbb{E}\left[\left(\frac{\eta_0}{a_n}\right)^2\mathbf{1}_{\{\eta_0\leq a_n\}}\right]\right),\\
\int_1^\infty\mathbb{P}\left(\eta_0\geq xa_n\right)\mathrm{d}x &= x\mathbb{P}\left(\eta_0\geq xa_n\right)\Big|_1^\infty+\int_1^\infty x\mathbb{P}\left(\frac{\eta_0}{a_n}\in\mathrm{d}x\right)
\end{aligned}$$

$$= -\mathbb{P}(\eta_0 \geq a_n) + \mathbb{E}\left[\frac{\eta_0}{a_n}\mathbf{1}_{\{\eta_0 \geq a_n\}}\right].$$

Using assumption (5.2) with $k = 1$ we have that

$$\lim_{n\to\infty} n \log\left(\mathbb{E}\left[e^{-\frac{\lambda}{a_n}(\eta_0 - \mathbb{E}[\eta_0])}\right]\right) = f(1)\lambda^\alpha$$

and therefore $\eta_0 - \mathbb{E}[\eta_0]$ belongs to the domain of attraction of the stable law with Laplace transform $e^{f(1)\lambda^\alpha}$. It then follows from (2.2), (2.3) and (2.4) that the sequences

$$n\mathbb{P}(\eta_0 \geq a_n), \quad n\mathbb{E}\left[\left(\frac{\eta_0}{a_n}\right)^2 \mathbf{1}_{\{\eta_0 \leq a_n\}}\right] \quad \text{and} \quad n\mathbb{E}\left[\frac{\eta_0}{a_n}\mathbf{1}_{\{\eta_0 \geq a_n\}}\right]$$

converge as $n \to \infty$. We therefore have that, for some constant C,

$$\mathbb{E}\left[\exp\left(-\frac{\lambda k}{a_n}\eta_0\right)\right] \leq 1 - \frac{\lambda k \mathbb{E}[\eta_0]}{a_n} + \frac{Ck^2}{n} \tag{5.7}$$

uniformly over $k \geq 1$.

We want to show that we can choose N_0 sufficiently large such that the right-hand side of (5.7) is strictly larger than zero for any $k \geq 1$ and $n \geq N_0$. Since $\alpha \in (1,2)$ we have that $n/a_n^2 \to 0$ as $n \to \infty$ therefore fix N_0 large enough such that, for $n \geq N_0$,

$$\frac{n(\lambda\mathbb{E}[\eta_0])^2}{Ca_n^2} < \frac{1}{2}.$$

We can write

$$1 - \frac{\lambda k \mathbb{E}[\eta_0]}{a_n} + \frac{Ck^2}{n} = 1 - \frac{k}{a_n}\left(\lambda\mathbb{E}[\eta_0] - \frac{Cka_n}{n}\right)$$

where the term in the brackets is only positive if $k < \lambda\mathbb{E}[\eta_0]n/(Ca_n)$. This means that

$$\frac{k}{a_n}\left(\lambda\mathbb{E}[\eta_0] - \frac{Cka_n}{n}\right) \leq \frac{\lambda\mathbb{E}[\eta_0]n}{Ca_n^2} \cdot \lambda\mathbb{E}[\eta_0] \leq \frac{1}{2}$$

for any $n \geq N_0$. This shows that the right-hand side of (5.7) is strictly larger than zero for any $k \geq 1$ and $n \geq N_0$.

Since $\log(1+x) \leq x$ for $x > -1$ we then have that

$$n \log\left(\mathbb{E}\left[\exp\left(-\frac{\lambda k}{a_n}\eta_0\right)\right]\right) \leq -\frac{n\lambda k \mathbb{E}[\eta_0]}{a_n} + Ck^2.$$

Combining this with (5.6) we then have that

$$n \log\left(\mathbf{E}\left[E^\omega\left[\exp\left(-\frac{\lambda}{a_n}(\eta_0 - \mathbb{E}[\eta_0])\right)\right]^k\right]\right) \leq Ck^2.$$

$\square$

We now show that the random variable χ_2 belongs to the domain of attraction of a stable law with index α. Since χ_j are i.i.d. and centred we then have a generalised central limit theorem for the sum of χ_j. Let $\mathcal{I} := \{Y_j\}_{j=\zeta_1^Y}^{\zeta_2^Y-1}$ denote the set of vertices in the regeneration block and for $k = 1, 2, ...$ let $\mathcal{I}_k := \{x \in \mathcal{I} : \ \mathcal{L}(x, \infty) = k\}$ denote those visited exactly k times.

Proposition 5.2.3. *Under the assumptions of Theorem 5.2*

$$\lim_{n\to\infty} n\mathbb{E}\left[\exp\left(-\frac{\lambda}{a_n}(\zeta_2^X - \zeta_1^X - \mathbb{E}[\eta_0](\zeta_2^Y - \zeta_1^Y))\right) - 1\right] = \lambda^\alpha \sum_{k=1}^{\infty} f(k)E[|\mathcal{I}_k|].$$

Proof. Since $\varrho_1 \geq 1$ we have that

$$E\left[|\mathcal{I}_k|\right] \ \leq \ E\left[\sum_{x\in\mathcal{I}} \mathbf{1}_{\{\mathcal{L}(x,\infty)\geq k\}}\right] \ = \ \sum_{x\in\mathbb{N}} E\left[\mathbf{1}_{\{x\in\mathcal{I}\}}\mathbf{1}_{\{\mathcal{L}(x,\infty)\geq k\}}\right].$$

Using Cauchy-Schwarz and that $\mathcal{L}(x, \infty)$ are identically distributed for $x \in \mathbb{N}$ we have that this is bounded above by

$$\sum_{x\in\mathbb{N}} P\left(x \in \mathcal{I}\right)^{1/2} P\left(\mathcal{L}(x,\infty) \geq k\right)^{1/2} \ = \ P\left(\mathcal{L}(0,\infty) \geq k\right)^{1/2} \sum_{x\in\mathbb{N}} P\left(x \in \mathcal{I}\right)^{1/2}.$$

The number of returns to the origin is geometrically distributed with probability u_∞ of moving away from the origin and never returning; therefore, $P\left(\mathcal{L}(0,\infty) \geq k\right) = (1 - u_\infty)^{k-1}$. By Lemma 2.3.7 we have that $P\left(x \in \mathcal{I}\right) \leq P(x \leq \zeta_2^Y) \leq Ce^{-cx}$. It follows that

$$E\left[|\mathcal{I}_k|\right] \ \leq \ C\left(1 - u_\infty\right)^{k/2} \sum_{x\geq 0} e^{-cx/2} \ \leq \ C\left(\frac{2}{\beta+1}\right)^{k/2}. \tag{5.8}$$

Conditional on Y, the holding times at different vertices are independent therefore

$$\begin{aligned}
&\mathbb{E}\left[\exp\left(-\frac{\lambda}{a_n}(\zeta_2^X - \zeta_1^X - \mathbb{E}[\eta_0](\zeta_2^Y - \zeta_1^Y))\right)\right] \\
&\qquad = \mathbb{E}\left[\exp\left(-\frac{\lambda}{a_n}\left(\sum_{x\in\mathcal{I}}\sum_{j=1}^{\mathcal{L}(x,\infty)}(\eta_{x,j} - \mathbb{E}[\eta_{x,j}])\right)\right)\right] \\
&\qquad = E\left[\prod_{x\in\mathcal{I}} \mathbb{E}^Y\left[\exp\left(-\frac{\lambda}{a_n}\left(\sum_{j=1}^{\mathcal{L}(x,\infty)}(\eta_{x,j} - \mathbb{E}[\eta_{x,j}])\right)\right)\right]\right].
\end{aligned}$$

Under $P^{\omega,Y}$ the holding times at a given vertex are independent, identically distributed and do not depend on Y; therefore, we can write the above expression as

$$E\left[\prod_{x\in\mathcal{I}}\mathbf{E}^Y\left[E^{\omega,Y}\left[\prod_{j=1}^{\mathcal{L}(x,\infty)} e^{-\frac{\lambda}{a_n}(\eta_{x,j}-\mathbb{E}[\eta_{x,j}])}\right]\right]\right]$$
$$= E\left[\prod_{k=1}^{\infty}\prod_{x\in\mathcal{I}_k}\mathbf{E}^Y\left[E^{\omega,Y}\left[e^{-\frac{\lambda}{a_n}(\eta_{x,1}-\mathbb{E}[\eta_{x,1}])}\right]^k\right]\right].$$

For $x \in \mathbb{Z}$ let $\tilde{\eta}_x = \eta_{x,1}$ be the first holding time at x. Notice that, since $\tilde{\eta}_x$ is independent of Y, we have that

$$\Psi_{k,x} := \mathbf{E}^Y\left[E^{\omega,Y}\left[e^{-\frac{\lambda}{a_n}(\eta_{x,1}-\mathbb{E}[\eta_{x,1}])}\right]^k\right] = \mathbf{E}\left[E^{\omega}\left[e^{-\frac{\lambda}{a_n}(\tilde{\eta}_x-\mathbb{E}[\tilde{\eta}_x])}\right]^k\right]$$

is independent of Y and, therefore, deterministic. In particular, it does not depend on x and we can write

$$\mathbb{E}\left[\exp\left(-\frac{\lambda}{a_n}\left(\zeta_2^X-\zeta_1^X-\mathbb{E}[\eta_0](\zeta_2^Y-\zeta_1^Y)\right)\right)\right] = E\left[\exp\left(\sum_{k=1}^{\infty}|\mathcal{I}_k|\log(\Psi_{k,0})\right)\right].$$

Furthermore, since $\tilde{\eta}_0 \geq 0$ we have that $\log(\Psi_{k,0}) \leq k\lambda\mathbb{E}[\eta_0]/a_n$.

Using a Taylor expansion we have that for any $x \in \mathbb{R}$ there exists $w \in [0,x]$ such that

$$e^x = 1 + x + \frac{e^w}{2}x^2.$$

It follows from this, and that $\log(\Psi_{k,0}) \leq k\lambda\mathbb{E}[\eta_0]/a_n$, that there exists a random variable W which is bounded above by

$$\frac{\lambda\mathbb{E}[\eta_0]}{a_n}\sum_{k=1}^{\infty}k|\mathcal{I}_k|$$

such that

$$nE\left[\exp\left(\sum_{k=1}^{\infty}|\mathcal{I}_k|\log(\Psi_{k,0})\right)-1\right]$$
$$= nE\left[\sum_{k=1}^{\infty}|\mathcal{I}_k|\log(\Psi_{k,0}) + \frac{e^W}{2}\left(\sum_{k=1}^{\infty}|\mathcal{I}_k|\log(\Psi_{k,0})\right)^2\right].$$

Using the previous upper bound on $\log(\Psi_{k,0})$ and Cauchy-Schwarz we then have that

$$nE\left[\frac{e^W}{2}\left(\sum_{k=1}^{\infty}|\mathcal{I}_k|\log(\Psi_{k,0})\right)^2\right] \leq \frac{C_\lambda n}{a_n^2}E\left[e^{\frac{\lambda\mathbb{E}[\eta_0]}{a_n}\sum_{k=1}^{\infty}k|\mathcal{I}_k|}\left(\sum_{k=1}^{\infty}k|\mathcal{I}_k|\right)^2\right]$$

$$\leq \frac{C_\lambda n}{a_n^2} E\left[e^{\frac{2\lambda\mathbb{E}[\eta_0]}{a_n}\sum_{k=1}^\infty k|\mathcal{I}_k|}\right]^{1/2} E\left[\left(\sum_{k=1}^\infty k|\mathcal{I}_k|\right)^4\right]^{1/2}.$$

The sum $\sum_{k=1}^\infty k|\mathcal{I}_k|$ is the time between regenerations of the embedded walk Y. By Lemma 2.3.7 this has exponential moments therefore we can choose n sufficiently large such that both of the expectations in the previous equation are finite. Since $a_n = n^{1/\alpha}L(n)$ for $\alpha \in (1,2)$ we have that n/a_n^2 converges to 0 as $n \to \infty$ and therefore

$$nE\left[\exp\left(\sum_{k=1}^\infty |\mathcal{I}_k|\log(\Psi_{k,0})\right) - 1\right] = nE\left[\sum_{k=1}^\infty |\mathcal{I}_k|\log(\Psi_{k,0})\right] + o(1).$$

The function $n\log(\Psi_{k,0})$ is deterministic and converges to the positive constant $f(k)\lambda$. By Lemma 5.2.2 we have that for n suitably large $ck \leq n\log(\Psi_{k,0}) \leq Ck^2$ independently of k. It follows from (5.8) that

$$\sum_{k=1}^\infty k^2 E\left[|\mathcal{I}_k|\right] < \infty,$$

and therefore by dominated convergence we have that

$$nE\left[\sum_{k=1}^\infty |\mathcal{I}_k|\log(\Psi_{k,0})\right] = \sum_{k=1}^\infty E\left[|\mathcal{I}_k|\right] n\log(\Psi_{k,0}) \to \lambda^\alpha \sum_{k=1}^\infty f(k)E\left[|\mathcal{I}_k|\right].$$

□

By Lemma 5.2.1 and Proposition 5.2.3 we now have that $\{\chi_j\}_{j\geq 2}$ are i.i.d. and belong to the domain of attraction of the stable law with Laplace exponent $\lambda^\alpha \nu_\beta \sum_{k=1}^\infty f(k)E[|\mathcal{I}_k|]$. We therefore have the following corollary.

Corollary 5.2.4. *Under the assumptions of Theorem 5.2, the sequence Σ_{nt}/a_n converges in distribution on $D_{J_1}([0,\infty),\mathbb{R})$ to a stable process $\tilde{\mathcal{V}}_t^\alpha$.*

We now have the required framework to prove Theorem 5.2 which is a convergence result for $(\Delta_{nt} - nt\nu_\beta^{-1})/a_n$ and $(X_{nt} - nt\nu_\beta)/a_n$. The approach we take is to compare Δ with Σ and then use a result of [76] that allows us to deduce convergence for the inverse of Δ.

Recall that for these processes we consider the M_1 topology as opposed to the stronger J_1 topology considered for Σ. Given that our approach is to compare Δ with Σ, one may question whether we should be able to extend the statements of Theorem 5.2 to the J_1 topology. This is not possible in the general setting because, between regenerations, the walk experiences several large holding times at the same significant vertex. This means that $\Delta_{nt} - nt\nu_\beta^{-1}$ fluctuates between regeneration points whereas Σ_m groups the fluctuations into a single jump.

Proof of Theorem 5.2. Write $m_t := \sup\{j \geq 0 : \zeta_j^X \leq t\}$ to be the number of regenerations by time t. As in the proof of Theorem 3.2, it follows by the law of large numbers and *continuity of the inverse at strictly increasing functions* (Proposition 2.1.1.v) that $m_{nt}n^{-1}$ converges $\mathbb{P}$-a.s. to $R_t := (\mathbb{E}[\eta_0]\mathbb{E}[\zeta_2^Y - \zeta_1^Y])^{-1}t$. It follows from Corollary 5.2.4 and J_1-*continuity of composition* (Proposition 2.1.1.ii) that $\Sigma_{m_{tn}}/a_n$ converges in distribution on $D_{J_1}([0,\infty),\mathbb{R})$ to the stable process $\tilde{\mathcal{V}}^\alpha_{R_t}$.

Recall that

$$\Sigma_m = \left(X_{S_{\zeta_m^Y}} - S_{\zeta_m^Y}\nu_\beta\right) - \left(\varrho_1 - \zeta_1^X\nu_\beta\right)$$

where $a_n^{-1}\left(\varrho_1 - \zeta_1^X\nu_\beta\right)$ converges to 0 in probability. It follows that

$$\Sigma_{m_{nt}} + \left(\varrho_1 - \zeta_1^X\nu_\beta\right) \;=\; X_{\zeta^X_{m_{nt}}} - \nu_\beta\zeta^X_{m_{nt}} \;=\; \nu_\beta\left(\varrho_{m_{nt}}\nu_\beta^{-1} - \Delta_{\varrho_{m_{nt}}}\right).$$

Notice that $\varrho_{m_{nt}}n^{-1}$ is bounded above by t and converges $\mathbb{P}$-a.s. to t since the distance between regenerations has exponential moments by Lemma 2.3.7. In particular, if

$$u_n := d_{M_1}\left(\left(\frac{\varrho_{m_{nt}}\nu_\beta^{-1} - \Delta_{\varrho_{m_{nt}}}}{a_n}\right)_{t\in[0,T]}, \left(\frac{nt\nu_\beta^{-1} - \Delta_{nt}}{a_n}\right)_{t\in[0,T]}\right)$$

converges to 0 in probability then $(\Delta_{nt} - nt\nu_\beta^{-1})/a_n$ converges in distribution on $D_{M_1}([0,\infty),\mathbb{R})$ to the stable process $\mathcal{V}^\alpha_t := -\nu_\beta^{-1}\tilde{\mathcal{V}}^\alpha_{R_t}$.

Since Δ_{nt} is increasing and the regeneration points are ordered we have that

$$\varrho_{m_{nt}} \leq nt \leq \varrho_{m_{nt}+1} \qquad \text{and} \qquad \Delta_{\varrho_{m_{nt}}} \leq \Delta_{nt} \leq \Delta_{\varrho_{m_{nt}+1}}.$$

Using that there are at most nT regenerations by level nT we then have that u_n is bounded above by $(1+\nu_\beta^{-1})\sup_{j\leq nT}(\varrho_{j+1} - \varrho_j)/a_n$. Let $\varepsilon > 0$, using a union bound and that the distance between regenerations have exponential moments we have that

$$\mathbb{P}\left(u_n > \varepsilon\right) \;\leq\; nTP(\varrho_2 - \varrho_1 > C\varepsilon a_n) + P(\varrho_1 > C\varepsilon a_n) \;\leq\; C_T n e^{-c_\varepsilon a_n}$$

which converges to 0 as $n \to \infty$.

Notice that the inverse $\Delta_t^{-1} = \sup\{X_s : \; s \leq t\} =: \overline{X}_t$ is the furthest vertex reached by time t. By *inverse with linear centring* (Proposition 2.1.1.iv) we have that $(\overline{X}_{nt} - nt\nu_\beta)/a_n$ converges in distribution on $D_{M_1}([0,\infty),\mathbb{R})$ to the stable process $\hat{\mathcal{V}}^\alpha_t := -\nu_\beta\mathcal{V}^\alpha_{\nu_\beta t}$. Moreover, using a union bound, we have that

$$\mathbb{P}\left(\sup_{t\leq nT}\overline{X}_t - X_t > C\log(n)\right) \;\leq\; nTP_{\lfloor C\log(n)\rfloor}(\tau_0^+ < \infty) \;=\; nT\beta^{-\lfloor C\log(n)\rfloor}$$

which converges to 0 for C large therefore we have the desired convergence for X. $\square$

Noting that $\hat{\mathcal{V}}_t^\alpha = \tilde{\mathcal{V}}^\alpha_{R_{\nu_\beta t}}$ is the limiting process for $(X_{nt} - nt\nu_\beta)/a_n$ where $R_t = (\mathbb{E}[\eta_0]\mathbb{E}[\zeta_2^Y - \zeta_1^Y])^{-1}t$ and $\tilde{\mathcal{V}}_t^\alpha$ has Laplace exponent $t\lambda^\alpha\nu_\beta\sum_{k=1}^\infty f(k)E[|\mathcal{I}_k|]$, we can express the Laplace transform of $\hat{\mathcal{V}}_t^\alpha$ as

$$\mathbb{E}[e^{-\lambda\hat{\mathcal{V}}_t^\alpha}] = \exp\left(\frac{t\lambda^\alpha\nu_\beta^2}{\mathbb{E}[\eta_0]\mathbb{E}[\zeta_2^Y - \zeta_1^Y]}\sum_{k=1}^\infty f(k)E[|\mathcal{I}_k|]\right). \tag{5.9}$$

We now prove a technical result that will make the condition (5.2) more straightforward to apply. We show that we can decompose the excursion times and remove parts of the excursion that have finite variance. Let X^+ be a randomly trapped random walk with some trap measure π^+ and embedded walk Y. Denote by η_k^+ the sequence of holding times and S_n^+ the clock process.

The idea is as follows. If we can couple X and X^+ so that $\mathbb{E}[(\eta_0 - \eta_0^+)^2] < \infty$ then, since the walks share an embedded walk and the environment is i.i.d., we can couple the two walks so that $\mathbb{E}[(\eta_k - \eta_k^+)^2] < \infty$ for all $k \in \mathbb{N}$. We then have that the difference between the two processes (suitably centred) is a process with no drift and increments which have finite variance. This degenerates under a_n scaling.

Corollary 5.2.5. *Suppose that the exists a coupling of X, X^+ such that $\mathbb{E}[(\eta_0 - \eta_0^+)^2] < \infty$. Let $a_n = n^{1/\alpha}L(n)$ for some $\alpha \in (1,2)$ and L slowly varying at ∞. If $\beta > 1$ and, for any $k \in \mathbb{N}$ and a real valued function $f \not\equiv 0$,*

$$n\log\left(\mathbf{E}\left[E^\omega\left[\exp\left(-\frac{\lambda}{a_n}(\eta_0^+ - \mathbb{E}[\eta_0^+])\right)\right]^k\right]\right) \sim f(k)\lambda^\alpha$$

as $n \to \infty$ then

1. *$(\Delta_{nt} - nt\nu_\beta^{-1})/a_n$ converges in $\mathbb{P}$-distribution as $n \to \infty$ on $D_{M_1}([0,\infty),\mathbb{R})$ to an α-stable process $\mathcal{V}_t^\alpha$.*

2. *$(X_{nt} - nt\nu_\beta)/a_n$ converges in $\mathbb{P}$-distribution as $n \to \infty$ on $D_{M_1}([0,\infty),\mathbb{R})$ to the α-stable process $-\nu_\beta\mathcal{V}^\alpha_{\nu_\beta t}$.*

Proof. Since, in each model, the traps are i.i.d. under $\mathbf{P}$ and each holding time in a fixed trap is independent there exists a version X^+ coupled to X such that $\mathbb{E}[(\eta_k - \eta_k^+)^2] < \infty$ for all $k \in \mathbb{N}$.

Let $\zeta_j^+ := \zeta_j^{X^+}$, $\zeta_j^- := \zeta_j^X - \zeta_j^+$ and $\eta_0^- := \eta_0 - \eta_0^+$ then we can write χ_j as

$$\left(\varrho_j - \varrho_{j-1} - \frac{\beta-1}{\beta+1}(\zeta_j^Y - \zeta_{j-1}^Y)\right)$$
$$+\,\nu_\beta\left(\mathbb{E}[\eta_0^+](\zeta_j^Y - \zeta_{j-1}^Y) - (\zeta_j^+ - \zeta_{j-1}^+)\right) + \nu_\beta\left(\mathbb{E}[\eta_0^-](\zeta_j^Y - \zeta_{j-1}^Y) - (\zeta_j^- - \zeta_{j-1}^-)\right).$$

It then follows from the proof of Theorem 5.2 that it suffices to show that

$$\max_{m\leq n}\sum_{j=2}^{m}\frac{\mathbb{E}[\eta_0^-](\zeta_j^Y-\zeta_{j-1}^Y)-(\zeta_j^- -\zeta_{j-1}^-)}{a_n}$$

converges to 0 in $\mathbb{P}$ probability. We have that $\mathbb{E}[\eta_0^-](\zeta_j^Y-\zeta_{j-1}^Y)-(\zeta_j^- -\zeta_{j-1}^-)$ are i.i.d. random variables with zero mean and finite variance therefore this follows from Doob's inequality since $a_n >> n^{1/2}$ similarly to the proof of Lemma 5.2.1. □

5.3 Applications

In this section we apply Theorems 5.1 and 5.2 to several classical models; namely, the continuous time random walk, the random walk in random scenery, the Bouchaud trap model, the transparent trap model and the comb model.

The continuous time random walk illustrates the behaviour when the environment is homogeneous; that is, the traps are identical. Much more is known for this model however we include it for its simplicity and to show that the results do not only hold for models where are majority of the slowing is caused by a few significant traps.

The random walk in random scenery is one of the simplest models of random walk in inhomogeneous random environment. In this model, each vertex is assigned a random holding time and the walk waits for this amount of time on every visit to the vertex. This simple structure is also a convenient counter-example which shows that, in general, Theorem 5.2 cannot be extended to J_1 convergence.

The remaining examples are classical models which give some indication of the behaviour we expect from random walks on GW-trees. In particular, the comb model is a model of a random walk on a random graph with a very similar structure to the subcritical GW-tree model. The graph consists of a collection of teeth attached to a fixed backbone. By choosing each tooth to be approximately geometric in length we have that the comb is equal in distribution the the subcritical GW-tree conditioned to survive where the foliage in the branches has been pruned; that is, the branches have been reduced to the single self-avoiding path to the deepest vertex.

5.3.1 Continuous time random walk

For the continuous time random walk we consider a non-random, homogeneous environment. Let ϖ be a $(0,\infty)$-valued probability measure. We then define the continuous time random walk as the randomly trapped random walk X with environment defined by $\omega_x=\varpi$ for all x. That is, $P^\omega(\eta_{x,j}\geq t)=\varpi([t,\infty))$ for all $x\in\mathbb{Z}$ and $j\geq 1$. Let S_n and Δ_n be its clock process and hitting times respectively.

Proposition 5.3.1 shows that the assumptions of Theorems 5.1 and 5.2 hold under a simple stable law condition for the holding times. In this case the results can be strengthened to the J_1 topology because the clock process becomes an i.i.d. sum however we do not show this here. It is also noteworthy that, because the environment is not random, convergence under the annealed law is equivalent to convergence under the quenched law.

Proposition 5.3.1. *Let $\alpha > 0$, $\beta > 1$ and suppose that $\overline{\omega}([t,\infty)) \sim t^{-\alpha}\overline{L}(t)$ as $t \to \infty$ for a function $\overline{L}$ which varies slowly at ∞.*

1. *If $\alpha \in (0,1)$ then*
   (a) *S_{nt}/a_n converges in $\mathbb{P}$-distribution as $n \to \infty$ on $D_{M_1}([0,\infty),\mathbb{R})$ to an α-stable subordinator $\mathcal{S}_t$;*
   (b) *X_{a_nt}/n converges in $\mathbb{P}$-distribution as $n \to \infty$ on $D_U([0,\infty),\mathbb{R})$ to the process $\frac{\beta-1}{\beta+1}\mathcal{S}_t^{-1}$.*
2. *If $\alpha \in (1,2)$ then, for a known constant $\nu_\beta = u_\infty/\mathbb{E}[\eta_0]$,*
   (a) *$(\Delta_{nt} - nt\nu_\beta^{-1})/a_n$ converges in $\mathbb{P}$-distribution as $n \to \infty$ on $D_{M_1}([0,\infty),\mathbb{R})$ to an α-stable process $\mathcal{V}_t^\alpha$.*
   (b) *$(X_{nt} - nt\nu_\beta)/a_n$ converges in $\mathbb{P}$-distribution as $n \to \infty$ on $D_{M_1}([0,\infty),\mathbb{R})$ to the α-stable process $-\nu_\beta\mathcal{V}_{\nu_\beta t}^\alpha$.*

Proof. For $\alpha \in (0,1)$, since the environment is not random, we have that η_0 is a non-negative random variable in the domain of attraction of a stable law with index α with respect to P^ω. In particular, there exists $a_n = n^{1/\alpha}L(n)$ for some L varying slowly at ∞ such that $nP^\omega(\eta_0 > ta_n) \sim ct^{-\alpha}$ and therefore

$$-n\log\left(\mathbf{E}\left[E^\omega\left[\exp\left(-\frac{\lambda}{a_n}\eta_0\right)\right]^k\right]\right) = -nk\log\left(E^\omega\left[\exp\left(-\frac{\lambda}{a_n}\eta_0\right)\right]\right) \sim Ck\lambda^\alpha.$$

The first result then follows from Theorem 5.1.

Similarly, for $\alpha \in (1,2)$, we have that $\eta_0 - \mathbb{E}[\eta_0]$ is a centred random variable which is bounded below and belongs to the domain of attraction of a stable law with index α with respect to P^ω. In particular, there exists $a_n = n^{1/\alpha}L(n)$ for some L varying slowly at ∞ such that $nP^\omega(\eta_0 - \mathbb{E}[\eta_0] > ta_n) \sim ct^{-\alpha}$ and therefore

$$n\log\left(\mathbf{E}\left[E^\omega\left[\exp\left(-\frac{\lambda}{a_n}(\eta_0 - \mathbb{E}[\eta_0])\right)\right]^k\right]\right) = nk\log\left(E^\omega\left[\exp\left(-\frac{\lambda}{a_n}(\eta_0 - \mathbb{E}[\eta_0])\right)\right]\right)$$

converges to $Ck\lambda^\alpha$ as $n \to \infty$. The result then follows from Theorem 5.2. $\square$

5.3.2 Random walk in random scenery

For the random walk in random scenery we have that all of the randomness comes from the environment and the embedded walk. Let $(\kappa_x)_{x\in\mathbb{Z}}$ be an i.i.d. sequence of $(0,\infty)$-valued random variables under $\mathbf{P}$. We then define the holding times by $P^\omega(\eta_{x,j} = \kappa_x) = 1$ for all $j \geq 1$ and $x \in \mathbb{Z}$. Let X_n denote the randomly trapped random walk, S_n its clock process and Δ_n the first hitting times.

Proposition 5.3.2 shows that the assumptions of Theorems 5.1 and 5.2 hold under a simple stable law condition for the environment.

Proposition 5.3.2. *Let $\alpha > 0$, $\beta > 1$ and suppose that $\mathbf{P}(\kappa_0 \geq t) \sim t^{-\alpha}\overline{L}(t)$ as $t \to \infty$ for a function $\overline{L}$ which varies slowly at ∞.*

1. *If $\alpha \in (0,1)$ then*
   (a) *S_{nt}/a_n converges in $\mathbb{P}$-distribution as $n \to \infty$ on $D_{M_1}([0,\infty),\mathbb{R})$ to an α-stable subordinator $\mathcal{S}_t$;*
   (b) *X_{a_nt}/n converges in $\mathbb{P}$-distribution as $n \to \infty$ on $D_U([0,\infty),\mathbb{R})$ to the process $\frac{\beta-1}{\beta+1}\mathcal{S}_t^{-1}$.*
2. *If $\alpha \in (1,2)$ then, for a known constant $\nu_\beta = u_\infty/\mathbb{E}[\eta_0]$,*
   (a) *$(\Delta_{nt} - nt\nu_\beta^{-1})/a_n$ converges in $\mathbb{P}$-distribution as $n \to \infty$ on $D_{M_1}([0,\infty),\mathbb{R})$ to an α-stable process $\mathcal{V}_t^\alpha$.*
   (b) *$(X_{nt} - nt\nu_\beta)/a_n$ converges in $\mathbb{P}$-distribution as $n \to \infty$ on $D_{M_1}([0,\infty),\mathbb{R})$ to the α-stable process $-\nu_\beta\mathcal{V}_{\nu_\beta t}^\alpha$.*

Proof. The holding time η_0 is almost surely fixed and equal to κ_0 under P^ω; we therefore have that $E^\omega[g(\eta_0)] = g(\kappa_0)$ for any function g.

For $\alpha \in (0,1)$, by assumption, κ_0 belongs to the domain of attraction of stable law of index α with respect to $\mathbf{P}$ hence there exists $a_n = n^{1/\alpha}L(n)$ such that $n\mathbf{P}(\kappa_0 > ta_n) \sim Ct^{-\alpha}$ and therefore, since κ_0 is almost surely non-negative,

$$-n\log\left(\mathbf{E}\left[E^\omega\left[\exp\left(-\frac{\lambda}{a_n}\eta_0\right)\right]^k\right]\right) = -n\log\left(\mathbf{E}\left[\exp\left(-\frac{k\lambda}{a_n}\kappa_0\right)\right]\right) \sim C(k\lambda)^\alpha.$$

The first result then follows from Theorem 5.1.

Similarly, for $\alpha \in (1,2)$, by assumption, κ_0 belongs to the domain of attraction of stable law of index α with respect to $\mathbf{P}$ hence there exists $a_n = n^{1/\alpha}L(n)$ such that $n\mathbf{P}(\kappa_0 - \mathbf{E}[\kappa_0] > ta_n) \sim Ct^{-\alpha}$ and therefore, since $\kappa_0 - \mathbf{E}[\kappa_0]$ is centred and bounded below,

$$n\log\left(\mathbf{E}\left[E^\omega\left[\exp\left(-\frac{\lambda}{a_n}(\eta_0 - \mathbb{E}[\eta_0])\right)\right]^k\right]\right) = n\log\left(\mathbf{E}\left[\exp\left(-\frac{k\lambda}{a_n}(\kappa_0 - \mathbb{E}[\kappa_0])\right)\right]\right)$$

converges to $C(k\lambda)^{\alpha}$ for some constant C. The result then follows from Theorem 5.2. $\square$

We now give an argument which shows that, in general, the M_1 convergence in Theorem 5.2 cannot be extended to J_1 convergence. For $f \in D([0,\infty),\mathbb{R})$ and $T,h>0$ let

$$\tilde{\omega}(f,T,h) := \inf_{(I_k)} \max_{k} \sup_{r,s\in I_k} |f(s)-f(r)|$$

denote the modulus of continuity where the infimum is over partitions of $[0,T)$ into $I_k=[v_k,v_{k+1})$ such that $v_{k+1}-v_k>h$ for all $k\geq 1$. By [48, Theorem 16.10], if $Z,(Z^n)_{n=1}^{\infty}$ are random elements of $D([0,\infty),\mathbb{R})$ then Z^n converges in distribution to Z in $D_{J_1}([0,\infty),\mathbb{R})$ if and only if Z^n converges to Z in the sense of finite dimensional distributions and, for any $T>0$,

$$\lim_{h\to 0}\limsup_{n\to\infty} \mathbb{E}[\tilde{\omega}(Z^n,T,h)\wedge 1]=0.$$

Let us assume that $\beta>1$ and $\mathbf{P}(\kappa_0\geq t)=t^{-\alpha}\wedge 1$ for $\alpha\in(1,2)$. Write

$$Z^n(t):=\frac{X_{tn}-tn\nu_\beta}{a_n} \qquad \text{then} \qquad |Z^n(s)-Z^n(r)|=\left|\frac{X_{sn}-X_{rn}-n\nu_\beta(s-r)}{a_n}\right|.$$

Since $\alpha>1$ we have that the walk is ballistic therefore

$$\lim_{n\to\infty}\mathbb{P}\left(\Delta_{nT\nu_\beta/2}\leq nT\right)=1.$$

That is, with high probability the walk visits at least $nT\nu_\beta/2$ vertices by time nT.

Let $\lambda>0$ then the probability that at least one of these first $nT\nu_\beta/2$ vertices has a holding time of at least $\lambda_n:=\lambda a_n=\lambda n^{1/\alpha}$ and no more than $h_n:=hn/3$ is

$$\mathbb{P}\left(\bigcup_{x=1}^{nT\nu_\beta/2}\{\kappa_x\in[\lambda_n,h_n)\}\right)=1-\left(1-(\lambda_n)^{-\alpha}+(h_n)^{-\alpha}\right)^{nT\nu_\beta/2}\to 1-e^{-\frac{\lambda^{-\alpha}T\nu_\beta}{2}}>0.$$

Let $\hat{x}:=\inf\{x\geq 0:\kappa_x\in[\lambda_n,h_n)\}$ then $\mathbb{P}(\kappa_{\hat{x}_n-1}\geq h_n)\to 0$ as $n\to\infty$ for any fixed $h>0$. Notice that the probability that the embedded walk moves from a vertex y to $y-1$ and then back to y starting from the first hitting time of y is

$$\mathbb{P}(Y_{\Delta_y^Y+1}=y-1,\ Y_{\Delta_y^Y+2}=y)=\frac{\beta}{(\beta+1)^2}>0$$

independently of the environment. In particular, this holds for $y=\hat{x}$ and, writing

$$\begin{aligned}\mathcal{A}_n:=\{\Delta_{nT\nu_\beta/2}\leq nT\}\,\cap\,\{\hat{x}\leq nT\nu_\beta/2\}\,\cap\,\{\kappa_{\hat{x}_n-1}<h_n\}\\ \cap\,\{Y_{\Delta_{\hat{x}}^Y+1}=\hat{x}-1\}\,\cap\,\{Y_{\Delta_{\hat{x}}^Y+2}=\hat{x}\},\end{aligned}$$

we have that for any fixed $h > 0$

$$\limsup_{n\to\infty} \mathbb{P}(\mathcal{A}_n) \;\geq\; \frac{\beta}{(\beta+1)^2}\left(1 - e^{-\frac{\lambda^{-\alpha}T\nu_\beta}{2}}\right) \;>\; 0.$$

Let $\vartheta_n := \Delta_{\hat{x}}$ be the first hitting time of $\hat{x}$ then $\vartheta_n^+ := \vartheta_n + \kappa_{\hat{x}}$ is the time that the walk first moves from $\hat{x}$ to one of its neighbours. Similarly, write $\tilde{\vartheta}_n := \inf\{t > \vartheta_n^+ : X_t = \hat{x}\}$ to be the first return time to $\hat{x}$ after the first visit and let $\tilde{\vartheta}_n^+ := \tilde{\vartheta}_n + \kappa_{\hat{x}}$ be the time at which the walk moves away from $\hat{x}$ for the second time.

Let (I_k) partition of $[0, T)$ into intervals $[v_k, v_{k+1})$ satisfying $v_{k+1} - v_k > h$ for all k. On the event $\mathcal{A}_n$ we have that for any such partition there exists some k such that either $\vartheta_n/n, \vartheta_n^+/n \in I_k$ or $\tilde{\vartheta}_n/n, \tilde{\vartheta}_n^+/n \in I_k$. We then have that

$$\limsup_{n\to\infty} |Z^n(\vartheta_n/n) - Z^n(\vartheta_n^+/n)| \;=\; \limsup_{n\to\infty} \frac{\nu_\beta \kappa_{\hat{x}} + 1}{a_n} \geq \lambda\nu_\beta.$$

The same holds for $\tilde{\vartheta}_n, \tilde{\vartheta}_n^+$. In particular, choosing $\lambda \leq \nu_\beta^{-1}$,

$$\limsup_{n\to\infty} \mathbb{E}[\tilde{\omega}(Z^n, T, h) \wedge 1] \;\geq\; \lambda\nu_\beta \limsup_{n\to\infty} \mathbb{P}(\mathcal{A}_n) \;\geq\; \frac{\lambda\nu_\beta\beta\left(1 - e^{-\frac{\lambda^{-\alpha}T\nu_\beta}{2}}\right)}{(\beta+1)^2} \;>\; 0.$$

This proves that

$$\lim_{h\to 0} \limsup_{n\to\infty} \mathbb{E}[\tilde{\omega}(Z^n, T, h) \wedge 1] \neq 0$$

for the random walk in random scenery and, therefore, Theorem 5.2 cannot be extended to $D_{J_1}([0,\infty), \mathbb{R})$.

5.3.3 Bouchaud trap model

Similarly to the random walk in random scenery, the environment in the Bouchaud trap model is defined by a one-parameter family $\kappa := (\kappa_x)_{x\in\mathbb{Z}}$ of i.i.d. $(0,\infty)$-valued random variables. For κ fixed we have that the holding times are exponentially distributed with mean κ_x at vertex x; that is $P^\omega(\eta_{x,j} \geq t) = e^{-t\kappa_x^{-1}}$ for $t \geq 0$.

Since η_0 is exponentially distributed under P^ω we have that, for $\lambda, a_n > 0$,

$$E^\omega\left[\exp\left(-\frac{\lambda}{a_n}\eta_0\right)\right] \;=\; \frac{1}{1 + \frac{\lambda\kappa_0}{a_n}} \;=\; 1 - \frac{\frac{\lambda\kappa_0}{a_n}}{1 + \frac{\lambda\kappa_0}{a_n}}. \tag{5.10}$$

The following lemma will, therefore, help to show that conditions (5.1) and (5.2) hold.

Lemma 5.3.3. *Let $a_n = n^{1/\alpha}L(n)$ for some $\alpha \in (0,1) \cup (1,2)$ and suppose that $n\mathbf{P}(\kappa_0 \geq ta_n) \sim ct^{-\alpha}$ as $n \to \infty$.*

1. *If $\alpha \in (0,1)$ then for any $j \geq 1$*

$$\lim_{n\to\infty} n\mathbf{E}\left[\left(\frac{\frac{\lambda\kappa_0}{a_n}}{1+\frac{\lambda\kappa_0}{a_n}}\right)^j\right] = \lambda^\alpha g(j) \tag{5.11}$$

for some function g.

2. *If $\alpha \in (1,2)$ then (5.11) holds for any $j \geq 2$. Moreover, for some constant C,*

$$\lim_{n\to\infty} n\left(\mathbf{E}\left[\frac{\lambda\kappa_0}{a_n}\right] - \mathbf{E}\left[\frac{\frac{\lambda\kappa_0}{a_n}}{1+\frac{\lambda\kappa_0}{a_n}}\right]\right) = C\lambda^\alpha.$$

Proof. To begin, note that, since $n\mathbf{P}(\kappa_0 \geq ta_n) \sim ct^{-\alpha}$, we have that $n\mathbf{P}(\lambda\kappa_0/a_n \in \mathrm{d}t)$ converges weakly to $c\lambda^\alpha t^{-(1+\alpha)}\mathrm{d}t$.

For $j \geq 1$ and $t \geq 0$ we have that

$$0 \leq \left(\frac{t}{1+t}\right)^j \leq 1 \wedge t^j \qquad \text{where} \qquad \int_0^\infty (1\wedge t^j)t^{-(1+\alpha)}\mathrm{d}t$$

exists whenever $\alpha \in (0,1)$ or $\alpha \in (1,2)$ and $j \geq 2$. In particular, it follows that

$$\lim_{n\to\infty} n\mathbf{E}\left[\left(\frac{\frac{\lambda\kappa_0}{a_n}}{1+\frac{\lambda\kappa_0}{a_n}}\right)^j\right] = c\lambda^\alpha \int_0^\infty \left(\frac{t}{1+t}\right)^j t^{-(1+\alpha)}\mathrm{d}t$$

which exists under either of these assumptions.

Similarly, for $\alpha \in (1,2)$ we have that

$$0 \leq t - \frac{t}{1+t} = \frac{t^2}{1+t} \leq t \wedge t^2 \qquad \text{where} \qquad \int_0^\infty (t\wedge t^2)t^{-(1+\alpha)}\mathrm{d}t$$

exists and therefore the second statement follows since we now have that

$$\lim_{n\to\infty} n\left(\mathbf{E}\left[\frac{\lambda\kappa_0}{a_n}\right] - \mathbf{E}\left[\frac{\frac{\lambda\kappa_0}{a_n}}{1+\frac{\lambda\kappa_0}{a_n}}\right]\right) = c\lambda^\alpha \int_0^\infty \frac{t^2}{1+t}t^{-(1+\alpha)}\mathrm{d}t.$$

□

Proposition 5.3.4 shows that the assumptions of Theorems 5.1 and 5.2 hold when κ_0 belong to the domain of attraction of a stable law.

Proposition 5.3.4. *Let $\beta > 1$, $a_n = n^{1/\alpha}L(n)$ for some $\alpha \in (0,1)\cup(1,2)$ and suppose that $n\mathbf{P}(\kappa_0 \geq ta_n) \sim ct^{-\alpha}$ as $n \to \infty$.*

1. *If $\alpha \in (0,1)$ then*

(a) S_{nt}/a_n *converges in* $\mathbb{P}$*-distribution as* $n \to \infty$ *on* $D_{M_1}([0,\infty),\mathbb{R})$ *to an* α*-stable subordinator* $\mathcal{S}_t$*;*

(b) $X_{a_n t}/n$ *converges in* $\mathbb{P}$*-distribution as* $n \to \infty$ *on* $D_U([0,\infty),\mathbb{R})$ *to the process* $\frac{\beta-1}{\beta+1}\mathcal{S}_t^{-1}$*.*

2. If $\alpha \in (1,2)$ *then, for a known constant* $\nu_\beta = u_\infty/\mathbb{E}[\eta_0]$*,*

(a) $(\Delta_{nt} - nt\nu_\beta^{-1})/a_n$ *converges in* $\mathbb{P}$*-distribution as* $n \to \infty$ *on* $D_{M_1}([0,\infty),\mathbb{R})$ *to an* α*-stable process* $\mathcal{V}_t^\alpha$*.*

(b) $(X_{nt} - nt\nu_\beta)/a_n$ *converges in* $\mathbb{P}$*-distribution as* $n \to \infty$ *on* $D_{M_1}([0,\infty),\mathbb{R})$ *to the* α*-stable process* $-\nu_\beta\mathcal{V}_{\nu_\beta t}^\alpha$*.*

Proof. For $\alpha \in (0,1)$, using (5.10) we have that

$$\mathbf{E}\left[E^\omega\left[\exp\left(-\frac{\lambda}{a_n}\eta_0\right)\right]^k\right] = 1 + \sum_{j=1}^k (-1)^j \binom{k}{j}\mathbf{E}\left[\left(\frac{\frac{\lambda\kappa_0}{a_n}}{1+\frac{\lambda\kappa_0}{a_n}}\right)^j\right].$$

By Lemma 5.3.3 it follows that

$$f(k) := -\lambda^{-\alpha}\lim_{n\to\infty} n\sum_{j=1}^k (-1)^j \binom{k}{j}\mathbf{E}\left[\left(\frac{\frac{\lambda\kappa_0}{a_n}}{1+\frac{\lambda\kappa_0}{a_n}}\right)^j\right]$$

exists and does not depend on λ. We then have that

$$-n\log\left(\mathbf{E}\left[E^\omega\left[\exp\left(-\frac{\lambda}{a_n}\eta_0\right)\right]^k\right]\right) = -n\log\left(1 - \frac{\lambda^\alpha f(k)}{n} + o(n^{-1})\right)$$

which converges as $n \to \infty$ to $\lambda^\alpha f(k)$. The first result then follows from Theorem 5.1.

For $\alpha \in (1,2)$, using (5.10) we have that

$$\mathbf{E}\left[E^\omega\left[\exp\left(-\frac{\lambda}{a_n}\eta_0\right)\right]^k\right] = 1 - k\mathbf{E}\left[\frac{\frac{\lambda\kappa_0}{a_n}}{1+\frac{\lambda\kappa_0}{a_n}}\right] + \sum_{j=2}^k (-1)^j \binom{k}{j}\mathbf{E}\left[\left(\frac{\frac{\lambda\kappa_0}{a_n}}{1+\frac{\lambda\kappa_0}{a_n}}\right)^j\right].$$

By Lemma 5.3.3 we then have that

$$f(k) := \lambda^{-\alpha}\lim_{n\to\infty} n\left(k\mathbf{E}\left[\frac{\lambda\kappa_0}{a_n}\right] - k\mathbf{E}\left[\frac{\frac{\lambda\kappa_0}{a_n}}{1+\frac{\lambda\kappa_0}{a_n}}\right] + \sum_{j=2}^k (-1)^j \binom{k}{j}\mathbf{E}\left[\left(\frac{\frac{\lambda\kappa_0}{a_n}}{1+\frac{\lambda\kappa_0}{a_n}}\right)^j\right]\right)$$

exists and does not depend on λ. In particular,

$$n\log\left(\mathbf{E}\left[E^\omega\left[\exp\left(-\frac{\lambda}{a_n}(\eta_0 - \mathbb{E}[\eta_0])\right)\right]^k\right]\right)$$

$$
\begin{aligned}
&= n\log\left(\left(1-k\frac{\lambda\mathbf{E}[\kappa_0]}{a_n}+\frac{\lambda^\alpha f(k)}{n}+o(n^{-1})\right)e^{\frac{k\lambda\mathbb{E}[\eta_0]}{a_n}}\right)\\
&= n\log\left(\left(1-k\frac{\lambda\mathbf{E}[\kappa_0]}{a_n}+\frac{\lambda^\alpha f(k)}{n}+o(n^{-1})\right)\left(1+k\frac{\lambda\mathbf{E}[\kappa_0]}{a_n}+o(n^{-1})\right)\right)\\
&= n\log\left(1+\frac{\lambda^\alpha f(k)}{n}+o(n^{-1})\right)
\end{aligned}
$$

since $\alpha \in (1,2)$ and $\mathbb{E}[\eta_0] = \mathbf{E}[\kappa_0]$. This converges as $n \to \infty$ to $\lambda^\alpha f(k)$. The result then follows from Theorem 5.2. □

5.3.4 Transparent trap model

The transparent trap model is an extension of the random walk in random scenery where the walk is able to skip large traps. It has been shown in [6] that, in the unbiased case, the limiting process can be either a linearly time changed Brownian motion, a fractional kinetics process or a FIN diffusion. In this section we determine the range of parameters for which the walk satisfies the assumptions of Theorem 5.2.

We now describe the model. For $\gamma > 0$ let $(\kappa_x)_{x\in\mathbb{Z}}$ be a sequence of i.i.d. $[1,\infty)$-valued random variables satisfying $\mathbf{P}(\kappa_x \geq t) \sim ct^{-\gamma}$ as $t \to \infty$ for a fixed constant c. For $\theta > 0$ fixed we define the environment at vertex x as $\omega_x := (1-\kappa_x^{-\theta})\delta_1 + \kappa_x^{-\theta}\delta_{\kappa_x}$ where δ denotes a Dirac measure. In this environment, the holding time at vertex x is κ_x with probability $\kappa_x^{-\theta}$ and a unit length of time otherwise. As usual, we let X denote the walk and Δ the first hitting times.

Proposition 5.3.5. *Suppose that $\alpha := \gamma+\theta \in (1,2)$ and $\beta > 1$; then, for $\nu_\beta = u_\infty/\mathbb{E}[\eta_0]$,*

1. *$(\Delta_{nt} - nt\nu_\beta^{-1})/a_n$ converges in $\mathbb{P}$-distribution as $n \to \infty$ on $D_{M_1}([0,\infty),\mathbb{R})$ to an α-stable process $\mathcal{V}_t^\alpha$.*

2. *$(X_{nt} - nt\nu_\beta)/a_n$ converges in $\mathbb{P}$-distribution as $n \to \infty$ on $D_{M_1}([0,\infty),\mathbb{R})$ to the α-stable process $-\nu_\beta\mathcal{V}_{\nu_\beta t}^\alpha$.*

Proof. Without loss of generality we may replace δ_1 with δ_0 in the definition of ω without affecting the result by Corollary 5.2.5. We then have that $\mathbb{E}[\eta_0] = \mathbf{E}[E^\omega[\eta_0]] = \mathbf{E}[\kappa_0^{1-\theta}]$ which is finite since $1-\theta < \gamma$.

By using the definition of ω_0 we have that

$$
\begin{aligned}
n\log&\left(\mathbf{E}\left[E^\omega\left[\exp\left(-\frac{\lambda}{n^{1/\alpha}}(\eta_0-\mathbb{E}[\eta_0])\right)\right]^k\right]\right)\\
&= \frac{n\lambda k\mathbb{E}[\eta_0]}{n^{1/\alpha}} + n\log\left(\mathbf{E}\left[\left(1-\kappa_0^{-\theta}+\kappa_0^{-\theta}e^{-\frac{\lambda\kappa_0}{n^{1/\alpha}}}\right)^k\right]\right)
\end{aligned}
$$

where

$$
\begin{aligned}
&\mathbf{E}\left[\left(1-\kappa_0^{-\theta}+\kappa_0^{-\theta}e^{-\frac{\lambda\kappa_0}{n^{1/\alpha}}}\right)^k\right]\\
&=1+\sum_{j=1}^k\binom{k}{j}\mathbf{E}\left[\kappa_0^{-\theta j}\left(e^{-\frac{\lambda\kappa_0}{n^{1/\alpha}}}-1\right)^j\right]\\
&=1-\frac{\lambda k\mathbf{E}[\kappa_0^{1-\theta}]}{n^{1/\alpha}}+k\mathbf{E}\Bigg[\kappa_0^{-\theta}\Bigg(e^{-\frac{\lambda\kappa_0}{n^{1/\alpha}}}-1+\frac{\lambda\kappa_0}{n^{1/\alpha}}\Bigg)\Bigg]+\sum_{j=2}^k\binom{k}{j}\mathbf{E}\Bigg[\kappa_0^{-\theta j}\Bigg(e^{-\frac{\lambda\kappa_0}{n^{1/\alpha}}}-1\Bigg)^j\Bigg].
\end{aligned}
$$

Using a Taylor expansion of $\log(x)$ and that $\mathbb{E}[\eta_0]=\mathbf{E}[\kappa_0^{1-\theta}]$ it now suffices to show that, for each $j\geq 2$,

$$
n\mathbf{E}\left[\kappa_0^{-\theta}\left(e^{-\frac{\lambda\kappa_0}{n^{1/\alpha}}}-1+\frac{\lambda\kappa_0}{n^{1/\alpha}}\right)\right]\qquad\text{and}\qquad n\mathbf{E}\left[\kappa_0^{-\theta j}\left(e^{-\frac{\lambda\kappa_0}{n^{1/\alpha}}}-1\right)^j\right]
$$

converge to $c\lambda^\alpha$ and $g(j)\lambda^\alpha$ for c,g not depending on λ.

For the first term we have that

$$
\begin{aligned}
n\mathbf{E}\left[\kappa_0^{-\theta}\left(e^{-\frac{\lambda\kappa_0}{n^{1/\alpha}}}-1+\frac{\lambda\kappa_0}{n^{1/\alpha}}\right)\right]&=\lambda^\theta n^{\gamma/\alpha}\mathbf{E}\left[\left(\frac{\lambda\kappa_0}{n^{1/\alpha}}\right)^{-\theta}\left(e^{-\frac{\lambda\kappa_0}{n^{1/\alpha}}}-1+\frac{\lambda\kappa_0}{n^{1/\alpha}}\right)\right]\\
&=\lambda^\theta n^{\gamma/\alpha}\int_0^\infty x^{-\theta}(e^{-x}-1+x)\mathbf{P}(\kappa_0\in n^{1/\alpha}\lambda^{-1}\mathrm{d}x)
\end{aligned}
$$

where, since $t^\gamma\mathbf{P}(\kappa_0\geq t)$ converges, we have that $n^{\gamma/\alpha}\mathbf{P}(\kappa_0\in n^{1/\alpha}\lambda^{-1}\mathrm{d}x)$ converges weakly to $c\lambda^\gamma x^{-(1+\gamma)}\mathrm{d}x$.

Notice that $0\leq|x^{-\theta}(e^{-x}-1+x)|\leq x^{-\theta}(x\wedge x^2)$ for $x\geq 0$. In particular, since $\theta+\gamma\in(1,2)$, we have that

$$
\int_0^\infty x^{-(1+\theta+\gamma)}(x\wedge x^2)\mathrm{d}x<\infty
$$

and, therefore, as $n\to\infty$,

$$
n\mathbf{E}\left[\kappa_0^{-\theta j}\left(e^{-\frac{\lambda\kappa_0}{n^{1/\alpha}}}-1+\frac{\lambda\kappa_0}{n^{1/\alpha}}\right)\right]\to c\lambda^{\theta+\gamma}\int_0^\infty x^{-(1+\theta+\gamma)}(e^{-x}-1+x)\mathrm{d}x.
$$

For $j\geq 2$, using a Taylor approximation we have that

$$
\begin{aligned}
0&\leq n\mathbf{E}\left[\kappa_0^{-\theta j}\left(1-e^{-\frac{\lambda\kappa_0}{n^{1/\alpha}}}\right)^j\right]\\
&=n\mathbf{E}\left[\kappa_0^{-\theta j}\left(1-e^{-\frac{\lambda\kappa_0}{n^{1/\alpha}}}\right)^j\mathbf{1}_{\{\kappa_0\leq\frac{n^{1/\alpha}}{\lambda}\}}\right]+n\mathbf{E}\left[\kappa_0^{-\theta j}\left(1-e^{-\frac{\lambda\kappa_0}{n^{1/\alpha}}}\right)^j\mathbf{1}_{\{\kappa_0>\frac{n^{1/\alpha}}{\lambda}\}}\right]
\end{aligned}
$$

$$\leq \lambda^j n^{1-\frac{j}{\alpha}} \mathbf{E}\left[\kappa_0^{j(1-\theta)} \mathbf{1}_{\{\kappa_0 \leq \frac{n^{1/\alpha}}{\lambda}\}}\right] + n\mathbf{E}\left[\kappa_0^{-\theta j} \mathbf{1}_{\{\kappa_0 > \frac{n^{1/\alpha}}{\lambda}\}}\right]. \tag{5.12}$$

If $\theta \geq 1$ then the first term in (5.12) converges to 0 because $\kappa_0^{j(1-\theta)} \leq 1$ and $j \geq 2 > \alpha$. If $\theta < 1$ then, using (2.2) and (2.4), we have that

$$n^{1-\frac{j}{\alpha}} \mathbf{E}\left[\kappa_0^{j(1-\theta)} \mathbf{1}_{\{\kappa_0 \leq \frac{n^{1/\alpha}}{\lambda}\}}\right] \sim C_\lambda n^{1-\frac{j}{\alpha}} n^{\frac{j-\theta j-\gamma}{\alpha}} = C_\lambda n^{-\frac{\theta(j-1)}{\alpha}}$$

which converges to 0 as $n \to \infty$ since $j \geq 2$.

Since $\theta > 0$ the second term in (5.12) is bounded above by

$$n\left(\frac{n^{1/\alpha}}{\lambda}\right)^{-\theta j} \mathbf{P}(\kappa_0 > n^{1/\alpha}\lambda^{-1}) = \lambda^{\theta j} n^{-\frac{\theta(j-1)}{\alpha}} n^{\gamma/\alpha} \mathbf{P}(\kappa_0 > n^{1/\alpha}\lambda^{-1})$$

which converges to 0 as $n \to \infty$ since $j \geq 2$ and $n^{\gamma/\alpha}\mathbf{P}(\kappa_0 > n^{1/\alpha}\lambda^{-1})$ converges. The result then follows from Theorem 5.2. $\square$

Recall that for the random walk in random scenery we had that $f(k) = Ck^\alpha$ and for the continuous time random walk $f(k) = Ck$. These are the two most extreme cases where all of the slowing is caused by the randomness in the environment (for the random walk in random scenery) or the randomness in the individual holding times (for the continuous time random walk). For the transparent traps model we have that $f(k) = Ck$. This corresponds to the environment appearing more homogeneous as a result of allowing the walk to ignore large traps in this manner.

5.3.5 Comb model

In this section we consider the comb model. This is a model of a random walk on a random graph in which we attach independent teeth to the vertices of $\mathbb{Z}$ (the spine). We then consider the biased random walk on this graph where the bias acts both along the backbone and into the teeth. This results in long excursions in the deepest teeth since the walk reaches the deepest vertex (apex) of the tooth with positive probability and then takes a large number of short excursions in the tooth before escaping to the spine. We only prove the analogue of Theorem 5.2 in this case.

Let $\mu \in (0,1)$ and $\mathcal{H} := (\mathcal{H}_x)_{x\in\mathbb{Z}}$ be an i.i.d. sequence of random variables taking values in $\mathbb{Z}^+$ and satisfying $\mathbf{P}(\mathcal{H}_x \geq n) \sim c_\mu \mu^n$ for some constant c_μ. The random variable $\mathcal{H}_x$ will denote the height of the tooth at x. For $\mathcal{H}$ fixed and $x \in \mathbb{Z}$ let $\mathcal{G}_x$ be the graph with vertices $\{x\}\times\{0,1,...,\mathcal{H}_x\}$ and undirected edges $\{((x,j-1),(x,j))\}_{j=1}^{\mathcal{H}_x}$. We then define the comb $\mathcal{G}$ as the graph formed by the concatenation of $\{\mathcal{G}_x\}_{x\in\mathbb{Z}}$ with additional edges $\{((x,0),(x+1,0))\}_{x\in\mathbb{Z}}$. We denote by $\mathcal{Y}$ the vertices $\mathbb{Z}\times\{0\}$ which form the spine. A segment of a comb with $\mathcal{H}_0 = 4$ can be seen in Figure 5.1.

Let d denote graph distance. For a fixed vertex x let $\overleftarrow{x}$ be $(x_1 - 1, 0)$ if $x = (x_1, 0)$ for some $x_1 \in \mathbb{Z}$ and $(x_1, x_2 - 1)$ if $x = (x_1, x_2)$ for some $x_1 \in \mathbb{Z}$ and $x_2 > 0$. That is, $\overleftarrow{x}$ is the vertex to the left of x if x is on the spine or below x otherwise. We then let $c(x) := \{y \in \mathcal{G} \setminus \overleftarrow{x} : d(x, y) = 1\}$ denote all other vertices of distance 1 from x. In Figure 5.1 we have that $c(x) = \{y, z\}$, $c(u) = \{v\}$ and we note that if x is the apex of a tooth then $c(x) = \emptyset$.

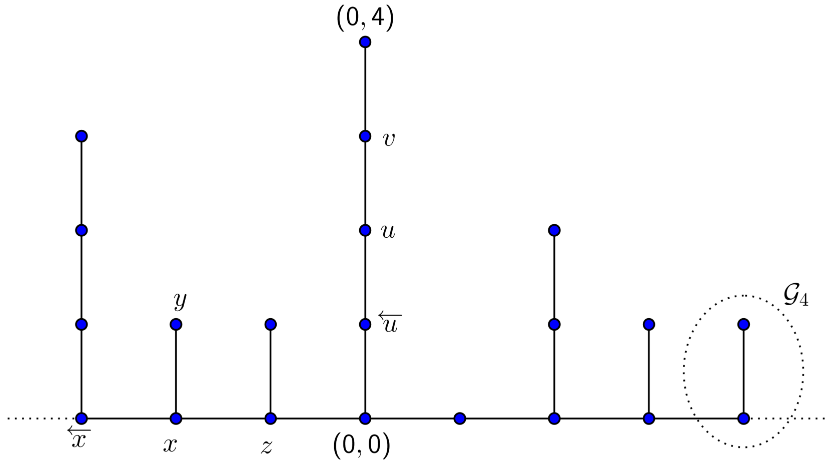

Figure 5.1: A section of a comb $\mathcal{G}$ with a tooth of height 4 emanating from $(0, 0)$.

We now define a biased random walk X on a fixed comb $\mathcal{G}$ to be the Markov chain started from $(0, 0)$ with transition probabilities:

$$P^{\mathcal{G}}(X_{n+1} = y | X_n = x) = \begin{cases} \frac{1}{1+\beta|c(x)|} & \text{if } y = \overleftarrow{x}, \\ \frac{\beta}{1+\beta|c(x)|} & \text{if } y \in c(x), \\ 0 & \text{otherwise.} \end{cases}$$

This walk is biased to move both along the spine and into the teeth. As the bias is increased the walk spends more time in the teeth which creates a slowing effect.

The main aim of this section is to prove Proposition 5.3.6 which determines the range of β and μ for which we observe stable fluctuations in the comb model. Due to a lattice effect we only observe convergence along subsequences. We write $|X_n| := d(X_n, (0, 0))$ to denote the distance between X_n and the starting point.

Proposition 5.3.6. *Suppose that $\alpha := \log(\mu^{-1})/\log(\beta) \in (1, 2)$ then, for any $\varsigma > 0$, denoting $n_l(\varsigma) := \lfloor \varsigma \mu^{-l} \rfloor$, we have that*

$$\frac{|\Delta_{n_l(\varsigma)t}| - n_l(\varsigma) t \nu_\beta^{-1}}{n_l(\varsigma)^{1/\alpha}} \qquad \textit{and} \qquad \frac{|X_{n_l(\varsigma)t}| - n_l(\varsigma) t \nu_\beta}{n_l(\varsigma)^{1/\alpha}}$$

converge in $\mathbb{P}$-distribution as $n \to \infty$ on $D_{M_1}([0,\infty),\mathbb{R})$ to α-stable processes.

With a slight abuse of notation we write n for $n_l(\varsigma)$ until we require the specific subsequences for the convergence to hold. We begin by simplifying the model to fit into the randomly trapped random walk framework. We then show that we can ignore certain parts of the excursion using Corollary 5.2.5. We exploit this by showing that η_0 can be approximated by a sum of i.i.d. excursions from the apex in the tooth. That is, we show that excursions which do not reach the apex are insignificant, as is the time taken to return from the apex to the root and the time taken to reach the apex started from the root. This allows us to reduce the model to a sum of excursions from the apex; we then reduce this further by replacing the excursions with i.i.d. excursions on an 'infinite tooth'.

To begin we note that, since $\mu \in (0,1)$ and $\alpha \in (1,2)$, we have that $\beta > 1$ therefore the bias is directed into the teeth and along the spine as opposed to being directed towards and along the spine. Also, note that the walk X can reach at most distance n from $(0,0)$ by time n; this means that it can see at most $n+1$ teeth by time n. The maximum distance from the spine is at most the height of the largest tooth seen therefore

$$\mathbb{P}\left(\sup_{m\leq Tn} d(X_m,\mathcal{Y}) > C\log(n)\right) \;\leq\; (\lceil Tn+1)\rceil \mathbf{P}(\mathcal{H}_0 > C\log(n)) \;\leq\; C_T n \mu^{C\log(n)}.$$

By choosing $C \geq 3/\log(\mu^{-1})$ we have that this is bounded above $C_T n^{-2}$ therefore, by the Borel-Cantelli lemma, we have that eventually the walk never deviates further than $C\log(n)$ from the spine up to time nT. In particular, since we consider polynomial scaling, this means that we can consider the walk projected onto the spine. This process is equal in distribution to a randomly trapped random walk on $\mathbb{Z}$ with holding times $\eta_{x,i}$ distributed as the time taken for the walk on the comb started at $(x,0)$ to reach either $(x-1,0)$ or $(x+1,0)$. With a slight abuse of notation we continue to use X to denote this randomly trapped random walk.

To ease future calculations we now introduce a simple model to describe the time spent in a tooth. For $m \geq 0$ let P^m denote the probability measure over the random walk Z on $\{-1,0,1,...,m\}$, started from 0, with transition probabilities $P^m(Z_{n+1}=0|Z_n=-1)=1$, $P^m(Z_{n+1}=m-1|Z_n=m)=1$ and for $j \notin \{-1,m\}$

$$P^m(Z_{n+1}=k|Z_n=j) = \begin{cases} \frac{1+\beta}{1+2\beta} & \text{if } j=0, k=-1, \\ \frac{\beta}{1+2\beta} & \text{if } j=0, k=1, \\ \frac{1}{1+\beta} & \text{if } j=k+1\neq 0, \\ \frac{\beta}{1+\beta} & \text{if } j=k-1\neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

We refer to m as the apex of the tooth, 0 as the root and -1 as the germ.

The sequence of heights $\mathcal{H}$ uniquely determines the environment ω. For $\mathcal{H}$ fixed let $(\eta_{x,j})_{x\in\mathbb{Z},j\geq 1}$ be independent with $\eta_{x,j}$ distributed as τ_{-1}^+ for the walk Z under $P_0^{\mathcal{H}_x}$ where $\tau_x^+ := \inf\{n>0 : Z_n = x\}$. The randomly trapped random walk with embedded walk Y (as the usual β-biased random walk on $\mathbb{Z}$) and holding times $(\eta_{x,j})_{x\in\mathbb{Z},j\geq 1}$ is then equal in distribution to the projection of the walk on the comb onto the spine.

We now wish to decompose the excursion times by removing parts of the excursion which are insignificant. If $\mathcal{H}_0 = 0$ then we have that $\eta_0 = 1$ deterministically so we ignore this case. Otherwise, from the root, the walk takes a geometric number G of excursions into the tooth before reaching the germ. For $x \in \{-1, 0, ..., \mathcal{H}_0\}$ we let $\tau_x^{(0)} := \tau_x^+$ and $\tau_x^{(k)} := \inf\{n > \tau_x^{(k-1)} : Z_n = x\}$ for $k \geq 1$ be the hitting times of x.

Let $\mathcal{A}_k := \{\tau_0^{(k)} > \inf\{n > \tau_0^{(k-1)} : Z_n = \mathcal{H}_0\}\}$ denote the event that the k^{th} excursion into the tooth reaches the apex. We denote by $\overline{T}_k := (\tau_0^{(k)} - \tau_0^{(k-1)})\mathbf{1}_{\mathcal{A}_k^c}$ the duration of the k^{th} excursion if the apex is not reached, $\overrightarrow{T}_k := (\inf\{n > \tau_0^{(k-1)} : Z_n = \mathcal{H}_0\} - \tau_0^{(k-1)})\mathbf{1}_{\mathcal{A}_k}$ the time taken to reach the apex when it is reached and $\overleftarrow{T}_k := (\tau_0^{(k)} - \sup\{n < \tau_0^{(k)} : Z_n = \mathcal{H}_0\})\mathbf{1}_{\mathcal{A}_k}$ the time taken to return from the apex (on the final excursion from the apex which does not return to the apex). Let $\tilde{T}_k := (\sup\{n < \tau_0^{(k)} : Z_n = \mathcal{H}_0\} - \inf\{n > \tau_0^{(k-1)} : Z_n = \mathcal{H}_0\})\mathbf{1}_{\mathcal{A}_k}$ be the time between the first and last hitting times of the apex on the k^{th} excursion. We then have that

$$\eta_0 = 1 + \sum_{k=1}^{G}\left(\tilde{T}_k + \overleftarrow{T}_k + \overrightarrow{T}_k + \overline{T}_k\right)$$

where, for $\mathcal{H}_0 \geq 1$,

$$P^{\mathcal{H}_0}(G = k) = \left(\frac{1+\beta}{1+2\beta}\right)\left(\frac{\beta}{1+2\beta}\right)^k.$$

Write

$$\eta_0^- = 1 + \sum_{k=1}^{G}\left(\overleftarrow{T}_k + \overrightarrow{T}_k + \overline{T}_k\right).$$

Lemma 5.3.7 shows that this term is insignificant.

Lemma 5.3.7. *Suppose that $\beta > 1$ and $\mu \in (0,1)$, then $\mathbb{E}[(\eta_0^-)^2] < \infty$.*

Proof. By Jensen's inequality we have that $\mathbb{E}[(\eta_0^-)^2] \leq 4\mathbb{E}[G^2]\mathbb{E}[1 + \overleftarrow{T}_1^2 + \overrightarrow{T}_1^2 + \overline{T}_1^2]$. Since G is geometrically distributed with parameter $\beta/(2\beta+1)$ we have that $\mathbb{E}[G^2] < \infty$.

For $\mathcal{H}_0 = 1$ we have that $\overrightarrow{T}_1 = \overleftarrow{T}_1 = 1$ and $\overline{T}_1 = 0$ deterministically. For $\mathcal{H}_0 > 1$ notice that

$$E^{\mathcal{H}_0}[\overline{T}_1^2] = E^{\mathcal{H}_0}[\overline{T}_1^2 | \tau_0^+ < \tau_{\mathcal{H}_0}^+]P_0^{\mathcal{H}_0}(\tau_0^+ < \tau_{\mathcal{H}_0}^+) \leq E_0^{\mathcal{H}_0}[(\tau_0^+)^2 | \tau_0^+ < \tau_{\mathcal{H}_0}^+]$$

and similarly

$$E^{\mathcal{H}_0}[\overleftarrow{T}_1^2] = E^{\mathcal{H}_0}[\overleftarrow{T}_1^2|\tau_0^+ < \tau_{\mathcal{H}_0}^+]P_{\mathcal{H}_0}^{\mathcal{H}_0}(\tau_0^+ < \tau_{\mathcal{H}_0}^+) \leq E_{\mathcal{H}_0}^{\mathcal{H}_0}[(\tau_0^+)^2|\tau_0^+ < \tau_{\mathcal{H}_0}^+].$$

In particular, we have that $E_0^{\mathcal{H}_0}[(\tau_0^+)^2|\tau_0^+ < \tau_{\mathcal{H}_0}^+] \leq E_{\mathcal{H}_0}[(\tau_0^+)^2|\tau_0^+ < \tau_{\mathcal{H}_0}^+]$ therefore, since $\mathbf{P}(\mathcal{H}_0 \geq n) \sim c_\mu \mu^n$ for some constant c_μ, we have that

$$\mathbb{E}[\overline{T}_1^2], \mathbb{E}[\overleftarrow{T}_1^2] \leq C\sum_{k=0}^{\infty} \mu^k E_k^k[(\tau_0^+)^2|\tau_0^+ < \tau_k^+].$$

We can write

$$\tau_0^+ = 1 + \sum_{j=1}^{k-1} v_j$$

where v_j is the number of visits to j before reaching 0. Each v_j is geometrically distributed with termination probability

$$P_j^k(\tau_0^+ < \tau_j^+|\tau_0^+ < \tau_k^+) = \frac{P_{j-1}^k(\tau_0^+ < \tau_j^+)}{(\beta+1)P_j^k(\tau_0^+ < \tau_k^+)} = \frac{(\beta-1)(\beta^k-1)}{(\beta+1)(\beta^j-1)(\beta^{k-j}-1)}$$

by Lemma 2.3.2. This is bounded below by $u_\infty := (\beta-1)/(\beta+1)$ independently of j, k. That is, each v_j is stochastically dominated by a geometric random variable with termination probability u_∞. It then follows from Jensen's inequality that

$$E_k^k[(\tau_0^+)^2|\tau_0^+ < \tau_k^+] \leq k^2 E\left[Geo\,(u_\infty)^2\right] \leq Ck^2. \tag{5.13}$$

Since $\mu \in (0,1)$ we therefore have that

$$\mathbb{E}[\overline{T}_1^2],\ \mathbb{E}[\overleftarrow{T}_1^2] \ \leq \ C\sum_{k=0}^{\infty} \mu^k E_k^k[(\tau_0^+)^2|\tau_0^+ < \tau_k^+] \ \leq \ C\sum_{k=0}^{\infty} \mu^k k^2 \ < \ \infty.$$

Similarly, we have that

$$\mathbb{E}[\overrightarrow{T}_1^2] \leq C\sum_{k=0}^{\infty} \mu^k E_0^k[(\tau_k^+)^2|\tau_k^+ < \tau_0^+].$$

In this case τ_k^+ can be written as

$$\tau_k^+ = 1 + \sum_{j=1}^{k-1} u_j$$

where u_j is the number of visits to j before reaching k. Each u_j is dominated by a geometric random variable with termination probability $1-\beta^{-1}$ by comparing it to the number of times an unconditioned β-biased random walk visits j. We therefore have that the method used for $\overleftarrow{T}_1$ also gives us that $\mathbb{E}[\overrightarrow{T}_1^2] < \infty$. $\square$

By Lemma 5.3.7 and Corollary 5.2.5 we can now consider $\eta_0^+ := \sum_{k=1}^G \tilde{T}_k$. We want to write this as an i.i.d. sum of excursions started from the apex conditioned to return to the apex. Conditional on G and $\mathcal{H}$ let $W \sim Bin(G, p_{\mathcal{H}_0})$ have the distribution of the number of excursions which reach the deepest point where

$$p_l := P_1(\tau_l < \tau_0) = \frac{\beta^{l-1}(\beta-1)}{\beta^l - 1}.$$

Then, define random variables $\tilde{G}_k$ such that $P^m(\tilde{G}_k = l) = p_{ret}(m)(1-p_{ret}(m))^l$ for $l \in \mathbb{Z}^+$ where, by Lemma 2.3.2, we have that the probability the walk returns to 0 from the apex m of the tooth without returning to m is

$$p_{ret}(m) := P_m^m(\tau_0^+ < \tau_m^+) = \frac{\beta-1}{\beta^m - 1}. \tag{5.14}$$

Let Z' be a random walk on $\mathbb{Z} \cap (-\infty, \mathcal{H}_0]$ started from $\mathcal{H}_0$, with transition probabilities

$$P^{\mathcal{H}_0}(Z'_{n+1} = k | Z'_n = j) = \begin{cases} \frac{1}{\beta+1} & \text{if } j = k+1 \neq \mathcal{H}_0, \\ \frac{\beta}{\beta+1} & \text{if } j = k-1, \\ 1 & \text{if } j = k+1 = \mathcal{H}_0, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, let Z'' be an independent copy of Z' conditioned on never hitting 0. Let $\hat{T} := \inf\{n > 0 : Z'_n = \mathcal{H}_0\}$ and $\tilde{\mathcal{A}} := \{\hat{T} < \inf\{n > 0 : Z'_n = 0\}\}$ be the first return time to the apex and the event that the walk returns to the apex before reaching 0. We then define $T' := \hat{T}\mathbf{1}_{\tilde{\mathcal{A}}} + \inf\{n > 0 : Z''_n = \mathcal{H}_0\}\mathbf{1}_{\tilde{\mathcal{A}}^c}$ be a random variable coupled to $\hat{T}$ such that T' is distributed as the duration of an excursion conditioned not to reach 0. Letting $(\hat{T}_{j,k}, T'_{j,k}, \tilde{\mathcal{A}}_{j,k})_{j,k\geq 1}$ be i.i.d. copies of $(\hat{T}, T', \tilde{\mathcal{A}})$ we have that

$$\eta_0^+ \stackrel{\mathrm{d}}{=} \sum_{k=1}^{W} \sum_{j=1}^{\tilde{G}_k} T'_{j,k}.$$

We want to replace each $T'_{j,k}$ with $\hat{T}_{j,k}$ so that we have a sum of i.i.d. random variables whose distribution does not depend on the height of the comb.

Lemma 5.3.8. *Suppose that $\beta > 1$ and $\beta\mu < 1$, then*

$$\mathbb{E}\left[\left(\sum_{k=1}^{W} \sum_{j=1}^{\tilde{G}_k} (T'_{j,k} - \hat{T}_{j,k})\right)^2\right] < \infty.$$

Proof. Using that $\mathbf{P}(\mathcal{H}_x \geq n) \sim c_\mu \mu^n$ for some constant c_μ and Jensen's inequality we have that

$$\begin{aligned}\mathbb{E}\left[\left(\sum_{k=1}^{W}\sum_{j=1}^{\tilde{G}_k}(T'_{j,k}-\hat{T}_{j,k})\right)^2\right] &\leq C\sum_{l=1}^{\infty}\mu^l E^l\left[\left(\sum_{k=1}^{W}\sum_{j=1}^{\tilde{G}_k}(T'_{j,k}-\hat{T}_{j,k})\mathbf{1}_{\tilde{\mathcal{A}}^c_{j,k}}\right)^2\right] \\ &\leq C\sum_{l=1}^{\infty}\mu^l E^l[W^2]E^l[\tilde{G}_1^2]E^l[(T'-\hat{T})^2\mathbf{1}_{\tilde{\mathcal{A}}^c}].\end{aligned}$$

Since $G \geq W$ is geometrically distributed with parameter $\beta/(2\beta+1)$ independently of l we have that $E^l[W^2] \leq \mathbb{E}[G^2] < \infty$. Since $\tilde{G}_1$ is geometrically distributed with parameter $1-p_{ret}(l)$ with respect to P^l we have that $E^l[\tilde{G}_k^2] \leq Cp_{ret}(l)^{-2}$. Notice that T' is stochastically dominated by $\hat{T}$ and is independent of $\tilde{\mathcal{A}}$. By Lemma 5.3.7 and (5.13) we then have that

$$\begin{aligned}E^l[(T'-T)^2\mathbf{1}_{\tilde{\mathcal{A}}^c}] &\leq P^l(\tilde{\mathcal{A}}^c)\left(E^l[(T')^2]+E^l[\hat{T}^2|\tilde{\mathcal{A}}^c]\right) \\ &\leq Cp_{ret}(l)\left(E^l_l[(\tau_l^+)^2|\tau_l<\tau_0^+]+E^l_l[(\tau_0^+)^2|\tau_0<\tau_l^+]+E^l_0[\tau_l^2]\right) \\ &\leq Cp_{ret}(l)l^2.\end{aligned}$$

Noting that $p_{ret}(l) \geq C\beta^{-l}$ for some constant C we have that

$$\begin{aligned}\mathbb{E}\left[\left(\sum_{k=1}^{G}\sum_{j=1}^{\tilde{G}_k}(T'_{j,k}-\hat{T}_{j,k})\right)^2\right] &\leq C\sum_{l=1}^{\infty}\mu^l E^l[G^2]E^l[\tilde{G}_k^2]E^l[(T'-\hat{T})^2\mathbf{1}_{\tilde{\mathcal{A}}^c}] \\ &\leq C\sum_{l=1}^{\infty}l^2(\beta\mu)^l\end{aligned}$$

which is finite since $\beta\mu < 1$. $\square$

By Lemma 5.3.8 we may now consider, instead of η_0,

$$\zeta := \sum_{k=1}^{W}\sum_{j=1}^{\tilde{G}_k}\hat{T}_{j,k}.$$

By comparison with a biased random walk on $\mathbb{Z}$ we have that $\hat{T}_{j,k}$ have exponential moments. More specifically, $\hat{T}_{j,k}-2$ can be stochastically dominated by the time taken for a biased random walk on $\mathbb{Z}$ to regenerate which has exponential moments by Lemma 2.3.7. Let $p_\infty := \lim_{l\to\infty} p_l = 1-\beta^{-1}$ be the probability that a β-biased walk started from 1 never visits 0. For W fixed we then define a random variable $B \sim Bin(W, p_\infty/p_{\mathcal{H}_0})$. Using that G has finite third moment and $1-p_\infty/p_l = \beta^{-l}$ we have that

$$\begin{aligned}
E^{\mathcal{H}_0}\Big[(W-B)^2\Big] &= \sum_{k=1}^{\infty} P(G=k) \sum_{j=1}^{k} P^{\mathcal{H}}(W=j|G=k) \sum_{i=1}^{j} P^{\mathcal{H}}(B=j-i|W=j) i^2 \\
&\leq \sum_{k=1}^{\infty} k^2 P(G=k) \sum_{j=1}^{k} P^{\mathcal{H}}(W=j|G=k) \sum_{i=1}^{j} P^{\mathcal{H}}(B=j-i|W=j) \\
&\leq \sum_{k=1}^{\infty} k^2 P(G=k) \sum_{j=1}^{k} P^{\mathcal{H}}(W=j|G=k) j \left(1-\frac{p_\infty}{p_{\mathcal{H}_0}}\right) \\
&\leq \left(1-\frac{p_\infty}{p_{\mathcal{H}_0}}\right) \sum_{k=1}^{\infty} k^3 P(G=k) \sum_{j=1}^{k} P^{\mathcal{H}}(W=j|G=k) \\
&\leq C\beta^{-\mathcal{H}_0}.
\end{aligned}$$

Using that $E^{\mathcal{H}_0}[\tilde{G}_k^2] \leq C\beta^{2\mathcal{H}_0}$ we then have that

$$\begin{aligned}
\mathbb{E}\left[\left(\sum_{k=1}^{W-B} \sum_{j=1}^{\tilde{G}_k} \hat{T}_{j,k}\right)^2\right] &\leq C \sum_{k=1}^{\infty} \mu^k E^{\mathcal{H}_0}\left[(W-B)^2\right] E^{\mathcal{H}_0}[\tilde{G}_k^2] E[T_{k,1}^2] \\
&\leq C \sum_{k=1}^{\infty} (\beta\mu)^k \\
&< \infty
\end{aligned}$$

since $\hat{T}_{j,k}$ have exponential moments. It follows that, by relabelling the excursions as $(\hat{T}_k)_{k\geq 1}$, we can instead consider

$$\zeta^+ := \sum_{k=1}^{D} \hat{T}_k \quad \text{where} \quad D := \sum_{j=1}^{B} \tilde{G}_j.$$

We are now able to complete the proof of Proposition 5.3.6 which shows that $(X_{nt} - nt\nu)/n^{1/\alpha}$ converges along subsequences.

Proof of Proposition 5.3.6. The number of excursions B is binomially distributed with G trials of success probability p_∞ where $G \sim Geo(\beta/(2\beta+1))$. By manipulating probability generating functions we have that B is geometrically distributed with

$$P(B=k) = \left(\frac{\beta-1}{2\beta}\right)^k \frac{\beta+1}{2\beta}$$

for $k \geq 0$. By Corollary 5.2.5, Lemmas 5.3.7, 5.3.8 and Theorem 5.2 it suffices to show

(5.2) with ζ^+ replacing η_0. Define

$$\varphi := \varphi_{\hat{T}}(\lambda n^{-1/\alpha}) = E\left[e^{-\frac{\lambda}{n^{1/\alpha}}\hat{T}_1}\right]$$

which is independent of the environment. We then have that

$$\mathbf{E}\left[E^{\mathcal{H}}\left[\exp\left(-\frac{\lambda}{n^{1/\alpha}}\zeta^+\right)\right]^k\right] = \mathbf{E}\left[E^{\mathcal{H}}\left[\varphi^D\right]^k\right] = \mathbf{E}\left[\left(\frac{\frac{\beta+1}{2\beta}}{1-\frac{\beta-1}{2\beta}\left(\frac{p_{ret}(\mathcal{H}_0)}{1-(1-p_{ret}(\mathcal{H}_0))\varphi}\right)}\right)^k\right].$$

Rearranging and substituting $p_{ret}(\mathcal{H}_0)$ from (5.14) we have that this is equal to

$$\mathbf{E}\left[\left(1-\frac{\beta-1}{2\beta}\cdot\frac{\psi}{1+\psi}\right)^k\right] \quad \text{where} \quad \psi = \frac{2\beta}{\beta^2-1}\left(\beta^{\mathcal{H}_0}-\beta\right)(1-\varphi).$$

We therefore have that

$$\begin{aligned}
&\mathbf{E}\left[E^{\mathcal{H}}\left[\exp\left(-\frac{\lambda}{n^{1/\alpha}}\zeta^+\right)\right]^k\right] \\
&\quad= \sum_{j=0}^{k}\binom{k}{j}\left(-\frac{\beta-1}{2\beta}\right)^j\mathbf{E}\left[\left(\frac{\psi}{1+\psi}\right)^j\right] \\
&\quad= 1-k\frac{\beta-1}{2\beta}\mathbf{E}\left[\frac{\psi}{1+\psi}\right]+\sum_{j=2}^{k}\binom{k}{j}\left(-\frac{\beta-1}{2\beta}\right)^j\mathbf{E}\left[\left(\frac{\psi}{1+\psi}\right)^j\right] \qquad (5.15)
\end{aligned}$$

where we note that

$$\mathbf{E}\left[\frac{\psi}{1+\psi}\right] = \mathbf{E}[\psi]-\mathbf{E}\left[\frac{\psi^2}{1+\psi}\right].$$

We want to show that $n\mathbf{E}[(\psi/(1+\psi))^j] \to \lambda^\alpha g(j)$ and $n\mathbf{E}[\psi^2/(1+\psi)] \to c\lambda^\alpha$ along subsequences $n_l(\varsigma)$ for some function g and any $j \geq 2$. By Lemma 5.3.3 it suffices to show that $n_l(\varsigma)\mathbf{P}(\psi \geq x) \sim C\lambda^\alpha x^{-\alpha}$.

Using a Taylor expansion and that $\hat{T}_1$ has finite second moments we have that $1-\varphi = \lambda E[\hat{T}_1]n^{-1/\alpha}+O(n^{-2/\alpha})$. By our assumption that $\mathbf{P}(\mathcal{H}_0 \geq n) \sim c_\mu \mu^n$, letting $\theta := \left(\frac{2\beta}{\beta^2-1}\lambda E[\hat{T}_1]\right)^{-1}$ and noting that $\alpha = \log(\mu^{-1})/\log(\beta)$ we have that

$$\begin{aligned}
n_l(\varsigma)\mathbf{P}(\psi \geq x) &= n_l(\varsigma)\mathbf{P}\left(\mathcal{H}_0 \geq \frac{\log\left(n_l(\varsigma)^{1/\alpha}x\theta\left(1+O(n_l(\varsigma)^{-1/\alpha})\right)\right)}{\log(\beta)}\right) \\
&= \lfloor\varsigma\mu^{-l}\rfloor\mathbf{P}\left(\mathcal{H}_0 \geq \frac{\lfloor\varsigma\mu^{-l}\rfloor}{\log(\mu^{-1})}+\frac{\log(x\theta)}{\log(\beta)}+O(l^{-1})\right)
\end{aligned}$$

which converges as $l \to \infty$ to

$$c_\mu \left(\frac{2\beta}{\beta^2 - 1} E[\hat{T}_1] \right)^\alpha x^{-\alpha} \lambda^\alpha \mathcal{I}$$

where $\mathcal{I} \in \{\mu, 1\}$.

We now return to proving that the convergence statement (5.2) holds. By a Taylor expansion we have that

$$e^{\frac{k\lambda}{n^{1/\alpha}} \mathbb{E}[\zeta^+]} = 1 + \frac{k\lambda}{n^{1/\alpha}} \mathbb{E}[\zeta^+] + o(n^{-1}) \tag{5.16}$$

where

$$\begin{aligned}
\frac{k\lambda}{n^{1/\alpha}} \mathbb{E}[\zeta^+] &= \frac{k\lambda}{n^{1/\alpha}} E[\hat{T}_1] \mathbf{E} \left[E^{\mathcal{H}}[D] \right] \\
&= \frac{k\lambda}{n^{1/\alpha}} E[\hat{T}_1] \mathbf{E} \left[E^{\mathcal{H}}[\tilde{G}_1] \right] E[B] \\
&= \frac{k\lambda}{n^{1/\alpha}} E[\hat{T}_1] \frac{\mathbf{E}[\beta^{\mathcal{H}_0}] - \beta}{\beta - 1} \frac{\beta - 1}{\beta + 1} \\
&= \frac{k\lambda}{n^{1/\alpha}} E[\hat{T}_1] \frac{\mathbf{E}[\beta^{\mathcal{H}_0}] - \beta}{\beta + 1}.
\end{aligned}$$

Moreover, since $\beta\mu < 1$, we have that $\mathbf{E}[\beta^{\mathcal{H}_0}] < \infty$. Since $\hat{T}_1$ has finite second moment, by a Taylor expansion we have that $1 - \varphi = \frac{\lambda}{n^{1/\alpha}} E[\hat{T}_1] + o(n^{-1})$ therefore

$$\begin{aligned}
k \frac{\beta - 1}{2\beta} \mathbf{E}[\psi] &= \frac{k\lambda}{n^{1/\alpha}} E[\hat{T}_1] \frac{\beta - 1}{2\beta} \frac{2\beta}{\beta^2 - 1} \left(\mathbf{E}\left[\beta^{\mathcal{H}_0}\right] - \beta \right) + o(n^{-1}) \\
&= \frac{k\lambda}{n^{1/\alpha}} E[\hat{T}_1] \frac{\mathbf{E}\left[\beta^{\mathcal{H}_0}\right] - \beta}{\beta + 1} + o(n^{-1}) \\
&= \frac{k\lambda}{n^{1/\alpha}} \mathbb{E}[\zeta^+] + o(n^{-1}).
\end{aligned} \tag{5.17}$$

In particular, it now follows from (5.15), (5.16), (5.17) that along subsequences $n_l(\varsigma)$ we have that

$$\begin{aligned}
&n \log \left(\mathbf{E} \left[E^{\mathcal{H}} \left[\exp \left(-\frac{\lambda}{n^{1/\alpha}} (\zeta^+ - \mathbb{E}[\zeta^+]) \right) \right]^k \right] \right) \\
&= n \log \left(\left(1 + \frac{k\lambda}{n^{1/\alpha}} \mathbb{E}[\zeta^+] + o(n^{-1}) \right) \left(1 - \frac{k\lambda}{n^{1/\alpha}} \mathbb{E}[\zeta^+] + \frac{\lambda^\alpha f(k)}{n} + o(n^{-1}) \right) \right) \\
&= \lambda^\alpha f(k) + o(1)
\end{aligned}$$

which proves that (5.2) holds along these subsequences. $\square$

Chapter 6

Random walks on supercritical Galton-Watson trees

In this chapter we consider the biased random walk on the supercritical GW-tree conditioned to survive $\mathcal{T}$ as described in Section 1.2.1. To recap, we only consider the case where the offspring law has deaths; that is, $p_0 > 0$. In this setting, a supercritical GW-tree conditioned to survive consists of an infinite backbone $\mathcal{Y}$ with finite trees attached as branches. The condition $p_0 > 0$ ensures that the tree has leaves (**P**-a.s.) and, therefore, the walk is slowed by trapping in the finite trees attached to the backbone. The backbone $\mathcal{Y}$ is itself a supercritical GW-tree whose offspring law does not have deaths.

As mentioned in Section 1.2.1 we prove two main results. We begin, in Section 6.1, by proving a quenched functional central limit theorem (Theorem 6.1) for the walk; that is, we prove conditions such that, for **P**-a.e. tree, the centred and rescaled walk converges in distribution to a Brownian motion. In Section 6.2 we consider the walk in the sub-ballistic regime. In this setting we show that the walk follows a polynomial escape regime but cannot be properly rescaled due to a lattice effect. This is Theorem 6.2.

Recall that for $\varsigma, t > 0$ and $n = 1, 2, ...$ we define

$$B_t^n := \frac{|X_{\lfloor nt \rfloor}| - n\nu_\beta t}{\varsigma\sqrt{n}}.$$

Our first main result of this chapter, Theorem 6.1, is a quenched invariance principle for B_t^n.

Theorem 6.1. *Suppose $p_0 > 0$, $\mu > 1$, $\beta \in (\mu^{-1}, f'(q)^{-1/2})$ and that there exists $\lambda > 1$ such that*

$$\sum_{k\geq 0} \lambda^k p_k < \infty. \tag{6.1}$$

Then, there exists $\varsigma > 0$ such that, for $\mathbf{P}$-a.e. $\mathcal{T}$, we have that the process $(B_t^n)_{t\geq 0}$ converges in $P^{\mathcal{T}}$-distribution on $D_{J_1}([0,\infty),\mathbb{R})$ to a standard Brownian motion.

To prove this we begin by proving an annealed functional central limit theorem (Corollary 6.1.3) for the walk by adapting the renewal argument used in [72, Theorem 4.1]. We then extend this to the quenched result by applying the argument used in [21] and [65] which largely involves showing that multiple copies of the walk see sufficiently different areas of the tree.

Our final result is an extension of [10, Theorems 1.1 & 1.3] for the walk in the sub-ballistic regime. Recall that $\Delta_n := \inf\{m \geq 0 : X_m \in \mathcal{Y}, |X_m| = n\}$, $\gamma := \log(n)/\log(f'(q)^{-1})$ and we now define the subsequences $n_l(t) := \lfloor tf'(q)^{-l}\rfloor$ for $t > 0$.

Theorem 6.2. *Suppose the offspring law belongs to the domain of attraction of a stable law of index $\alpha \in (1,2)$, has mean $\mu > 1$ and the bias satisfies the bound $\beta > f'(q)^{-1}$. Then,*

$$\frac{\Delta_{n_l(t)}}{n_l(t)^{\frac{1}{\gamma}}} \to R_t$$

in distribution as $l \to \infty$ under $\mathbb{P}$, where R_t is a random variable with an infinitely divisible law whose parameters are given in [10, Theorem 1.4]. Moreover, the laws of $(\Delta_n n^{-\frac{1}{\gamma}})_{n\geq 0}$ and $(|X_n|n^{-\gamma})_{n\geq 0}$ under $\mathbb{P}$ are tight on $(0,\infty)$ and $\mathbb{P}$-a.s.

$$\lim_{n\to\infty} \frac{\log|X_n|}{\log(n)} = \gamma.$$

In [10], the case in which the offspring law has finite variance is considered. The argument we use for the infinite variance case is generally the same as in the finite variance case but needs some technical input. It also follows similarly to Theorems 4.1, 4.2 and 4.3 therefore we will only sketch the proof.

6.1 Functional central limit theorems

The aim of this section is to prove Theorem 6.1. We assume that the exponential moments condition (6.1) holds throughout. This is a purely technical assumption which we expect could be relaxed to a sufficiently large moment condition however the main focus of this work has been to obtain this upper bound on the bias which we believe to be optimal.

We begin this section by showing that the time spent in a branch has finite variance. Recall that g, h are the probability generating functions for the GW-processes associated with the backbone and the traps respectively. Let $\overline{\mathcal{T}}^h$ be the tree formed by attaching an additional vertex $\overline{\rho}$ (as the parent of the root ρ) to an h-GW-tree $\mathcal{T}^h$. For a fixed tree $\mathscr{T}$ and vertex $x \in \mathscr{T}$ let $\tau_x^+ := \inf\{k > 0 : X_k = x\}$ denote the first return time to x. Let ξ^f, ξ^g, ξ^h be random variables with probability generating functions f, g and h respectively then let ξ be equal in distribution to the number of vertices in the first generation of $\mathcal{T}$. Since the generation sizes of $\mathcal{T}^g$ are dominated by those of $\mathcal{T}$ we have that ξ^g is stochastically dominated by ξ. Using Bayes' law we have that $\mathbf{P}(\xi = k) = p_k(1-q^k)(1-q)^{-1} \leq cp_k$ therefore both ξ and ξ^g inherit the exponential moments of ξ^f. Furthermore $\mathbf{P}(\xi^h = k) = p_k q^k$ therefore ξ^h automatically has exponential moments.

Lemma 6.1.1. *Under the assumptions of Theorem 6.1 we have that*

$$\mathbf{E}\left[E_{\overline{\rho}}^{\overline{\mathcal{T}}^h}\left[\left(\tau_{\overline{\rho}}^+\right)^2\right]\right] < \infty.$$

Proof. We can write

$$\tau_{\overline{\rho}}^+ = \sum_{x \in \overline{\mathcal{T}}^h} v_x \qquad \text{where} \qquad v_x = \sum_{k=1}^{\tau_{\overline{\rho}}^+} \mathbf{1}_{\{X_k = x\}}$$

is the number of visits to x before returning to $\overline{\rho}$. Recall that $c(x)$ denotes the set of children of x. It then follows that

$$\begin{aligned}
\mathbf{E}\left[E_{\overline{\rho}}^{\overline{\mathcal{T}}^h}\left[\left(\tau_{\overline{\rho}}^+\right)^2\right]\right] &= \mathbf{E}\left[\sum_{x,y \in \overline{\mathcal{T}}^h} E_{\overline{\rho}}^{\overline{\mathcal{T}}^h}[v_x v_y]\right] \\
&\leq C_\beta \mathbf{E}\left[\sum_{x,y \in \overline{\mathcal{T}}^h} (|c(x)|\beta + 1)(|c(y)|\beta + 1)\beta^{|x|+|y|}\right] \\
&= C_\beta \mathbf{E}\left[\sum_{x \in \overline{\mathcal{T}}^h} (|c(x)|\beta + 1)\beta^{|x|} \sum_{y \in \overline{\mathcal{T}}^h} (|c(y)|\beta + 1)\beta^{|y|}\right] \qquad (6.2)
\end{aligned}$$

where the inequality follows from Lemma 2.3.6. Letting Z_k^h denote the size of the k^{th} generation of $\overline{\mathcal{T}}^h$ and collecting terms in each generation we have that

$$\sum_{x \in \overline{\mathcal{T}}^h} (|c(x)|\beta + 1)\beta^{|x|} \; = \; 1 + 2\sum_{k \geq 1} Z_k^h \beta^k.$$

By Lemma 2.4.1, since ξ^h has exponential moments, we have that $\mathbf{E}[Z_k^h Z_j^h] \leq Cf'(q)^j$ whenever $j \geq k$. Substituting this and the above inequality into (6.2) we have that

$$\begin{aligned}
\mathbf{E}\left[E_{\overline{\rho}}^{\overline{\mathcal{T}}^h}\left[\left(\tau_{\overline{\rho}}^+\right)^2\right]\right] &\leq C_\beta \sum_{k \geq 0} \beta^k \sum_{j \geq k} \mathbf{E}[Z_k^h Z_j^h]\beta^j \\
&\leq C_\beta \sum_{k \geq 0} \beta^k \sum_{j \geq k} (f'(q)\beta)^j
\end{aligned}$$

$$\leq C_{\beta,f'(q)} \sum_{k\geq 0}(f'(q)\beta^2)^k$$

which is finite by the assumption that $\beta < f'(q)^{-1/2}$. □

Let $S_0 := 0$, $S_n := \inf\{k > S_{n-1} : X_k, X_{k-1} \in \mathcal{T}^g\}$ for $n \geq 1$ and $Y_n := X_{S_n}$, then Y_n is a β-biased random walk on $\mathcal{T}^g$ coupled to X_n. Write $\zeta_0^Y := 0$ and for $m = 1, 2, ...$ let

$$\zeta_m^Y := \inf\{k > \zeta_{m-1}^Y : |Y_j| < |Y_k| \leq |Y_l| \text{ for all } j < k \leq l\}$$

be regeneration times for the backbone walk. We can then define $\varrho_k := Y_{\zeta_k^Y}$ to be the regeneration points and $\zeta_k^X := \inf\{m \geq 0 : X_m = \varrho_k\}$ to be the corresponding regeneration times for X. By [55, Proposition 3.4] we have that there exists, **P**-a.s., an infinite sequence of regeneration times $\{\zeta_k^X\}_{k\geq 1}$ and the sequence

$$\left\{\left(\zeta_{k+1}^X - \zeta_k^X\right), \left(|\varrho_{k+1}| - |\varrho_k|\right)\right\}_{k\geq 1}$$

is i.i.d. (as is the corresponding sequence for Y). Furthermore, letting $m_t := \sup\{j \geq 0 : \zeta_j^X \leq t\}$ be the number of regenerations by time t, we have that m_t is non-decreasing and diverges $\mathbb{P}$-a.s.

By [55, Theorems 3.1 & 4.1], whenever $\mu > 1$ and $\mu^{-1} < \beta < f'(q)^{-1}$ we have that there exists $\nu_\beta \in (0, 1)$ such that $|X_n|n^{-1}$ converges $\mathbb{P}$-a.s. to ν_β. Moreover, combined with [55, Corollary 3.5], we have that the time and distance between regenerations of X both have finite means with respect to $\mathbb{P}$. Let

$$\chi_j := |\varrho_j| - |\varrho_{j-1}| - \nu_\beta(\zeta_j^X - \zeta_{j-1}^X).$$

By the previous remark we have that χ_j are i.i.d. with respect to $\mathbb{P}$. By the strong law of large numbers and the definition of ν_β we have that χ_j are centred (see [55, Theorems 3.1 & 4.1]). We will show that χ_j have finite second moment and that their sum

$$\Sigma_m := \sum_{j=2}^{m} \chi_j = \left(\left|X_{\zeta_m^X}\right| - \nu_\beta\zeta_m^X\right) - \left(\left|X_{\zeta_1^X}\right| - \nu_\beta\zeta_1^X\right)$$

can be used to approximate B_t^n.

By the remark preceding Lemma 6.1.1, the offspring distribution ξ^g of $\mathcal{T}^g$ has exponential moments. Since Y is a random walk on $\mathcal{T}^g$, by [65, Proposition 3] we have that $\mathbb{E}[(\zeta_2^Y - \zeta_1^Y)^k] < \infty$ for all $k \in \mathbb{Z}$ whenever $\beta > \mu^{-1}$.

Let $\eta_k := S_{k+1} - S_k$ denote the total time taken between X making the k^{th}

and $(k+1)^{\text{th}}$ transition along the backbone. This time consists of

$$N_k := \sum_{j=S_k+1}^{S_{k+1}} \mathbf{1}_{\{X_j = Y_k\}}$$

excursions into the finite trees appended to the backbone at this vertex and one additional step to the next backbone vertex. Write $\tau_n^{(0)} := S_n$ and $\tau_n^{(j)} := \inf\{k > \tau_n^{(j-1)} : X_k = Y_n\}$ for $j \geq 1$ to be the hitting times of the backbone after time S_n. We can then write

$$\eta_k := 1 + \sum_{j=1}^{N_k} \gamma_{k,j} \qquad \text{where} \qquad \gamma_{k,j} := \tau_k^{(j)} - \tau_k^{(j-1)} \tag{6.3}$$

is the duration of the j^{th} such excursion.

Proposition 6.1.2. *Under the assumptions of Theorem 6.1 we have that*

$$\mathbb{E}[(\zeta_2^X - \zeta_1^X)^2] < \infty.$$

Proof. The law of $\zeta_2^X - \zeta_1^X$ under $\mathbb{P}$ is identical to its law under $\mathbb{P}_\rho(\cdot|\zeta_1^Y = 1)$. That is, by the independence structure, we can condition the first regeneration vertex to be the first vertex reached by Y without changing the law of $\zeta_2^X - \zeta_1^X$. We therefore have that $\mathbb{E}\left[(\zeta_2^X - \zeta_1^X)^2\right]$ can be written as

$$\mathbb{E}\big[(\zeta_2^X - \zeta_1^X)^2\big|\zeta_1^Y = 1\big] = \mathbb{E}\left[\left(\sum_{k=1}^{\zeta_2^Y - \zeta_1^Y} \eta_k\right)^2 \Bigg|\zeta_1^Y = 1\right] \leq \mathbb{E}\left[(\zeta_2^Y - \zeta_1^Y)\sum_{k=1}^{\zeta_2^Y - \zeta_1^Y} \eta_k^2\Big|\zeta_1^Y = 1\right]$$

by convexity. Using convexity again with the decomposition (6.3) we can write this as

$$\begin{aligned}\mathbb{E}&\left[(\zeta_2^Y - \zeta_1^Y)\sum_{k=1}^{\zeta_2^Y - \zeta_1^Y}\left(1 + \sum_{j=1}^{N_k}\gamma_{k,j}\right)^2 \Bigg|\zeta_1^Y = 1\right] \\ &\qquad \leq \mathbb{E}\left[(\zeta_2^Y - \zeta_1^Y)\sum_{k=1}^{\zeta_2^Y - \zeta_1^Y}(N_k + 1)\left(1 + \sum_{j=1}^{N_k}\gamma_{k,j}^2\right)\Bigg|\zeta_1^Y = 1\right].\end{aligned}$$

The excursion times $\gamma_{k,j}$ are distributed as the first return time to $\overline{\rho}$ for a walk started from $\overline{\rho}$ on $\overline{\mathcal{T}}^h$. Moreover, under $\mathbb{P}$, they are independent of the backbone, the buds and the walk on the backbone and buds. In particular, they are independent of the regeneration times of Y and the number of excursions therefore we have that the above expectation can be bounded above by

$$\mathbf{E}\left[E_{\overline{\rho}}^{\mathcal{T}^h}\left[(\tau_{\overline{\rho}}^+)^2\right]\right]\mathbb{E}\left[(\zeta_2^Y-\zeta_1^Y)\sum_{k=1}^{\zeta_2^Y-\zeta_1^Y}(N_k+1)^2\Big|\zeta_1^Y=1\right].$$

Where, by Lemma 6.1.1, we have that $\mathbf{E}\left[E_{\overline{\rho}}^{\mathcal{T}^h}\left[(\tau_{\overline{\rho}}^+)^2\right]\right]<\infty$.

Let $(z_j)_{j=0}^\infty$ denote the ordered distinct vertices visited by Y and

$$\mathcal{L}(z,j):=\sum_{i=0}^j \mathbf{1}_{\{Y_j=z\}},\quad \mathcal{L}(z):=\mathcal{L}(z,\infty)$$

the local times of the vertex z. Write

$$W_{z,l}:=\sum_{j=0}^\infty \mathbf{1}_{\{X_j=z,\,X_{j+1}\notin\mathcal{T}^g,\,\mathcal{L}(z,j)=l\}}$$

to be the number of excursions from z (by X) on the l^{th} visit to z (by Y) for $l=1,...,\mathcal{L}(z)$ and $M:=|\{Y_k\}_{k=1}^{\zeta_2^Y-1}|$ the number of distinct vertices visited by Y between time 1 and time ζ_2^Y-1 then

$$\begin{aligned}
&\mathbb{E}\left[(\zeta_2^Y-\zeta_1^Y)\sum_{k=1}^{\zeta_2^Y-\zeta_1^Y}(N_k+1)^2\Big|\zeta_1^Y=1\right] \qquad (6.4)\\
&\quad=\mathbb{E}\left[(\zeta_2^Y-\zeta_1^Y)\sum_{k=1}^{M}\sum_{l=1}^{\mathcal{L}(z_k)}(W_{z_k,l}+1)^2\Big|\zeta_1^Y=1\right]\\
&\quad=\sum_{k=1}^\infty\sum_{l=1}^\infty \mathbb{E}\left[(\zeta_2^Y-\zeta_1^Y)\mathbf{1}_{\{k\le M,\,l\le\mathcal{L}(z_k)\}}(W_{z_k,l}+1)^2|\zeta_1^Y=1\right]\\
&\quad\le\sum_{k=1}^\infty\sum_{l=1}^\infty\left(\mathbb{E}\left[(\zeta_2^Y-\zeta_1^Y)^2\mathbf{1}_{\{k\le M,\,l\le\mathcal{L}(z_k)\}}|\zeta_1^Y=1\right]\mathbb{E}\left[(W_{z_k,l}+1)^4|\zeta_1^Y=1\right]\right)^{1/2}
\end{aligned}$$

by the Cauchy-Schwarz inequality. Conditional on $\zeta_1^Y=1$, for all $1\le k\le M$ we have that $\mathcal{L}(z_k)\le\zeta_2^Y-\zeta_1^Y$; moreover, $M\le\zeta_2^Y-\zeta_1^Y$ therefore

$$\mathbf{1}_{\{k\le M,\,l\le\mathcal{L}(z_k)\}}\le\mathbf{1}_{\{k,l\le\zeta_2^Y-\zeta_1^Y\}}.$$

Since the root does not have a parent, without any further information concerning the number of children from a given vertex, we have that the walk is more likely to take an excursion into one of the neighbouring traps when at the root than from this vertex. We can, therefore, stochastically dominate the number of excursions from a vertex by the number of excursions from the root to see that $\mathbb{E}\left[(W_{z_k,l}+1)^4\right]\le\mathbb{E}\left[(W_{z_0,1}+1)^4\right]$. Using this, the Cauchy-Schwarz inequality and that $\mathbb{P}(\zeta_1^Y = 1) > 0$, the expression (6.4) can be bounded above by

$$\begin{aligned}\mathbb{P}(\zeta_1^Y = 1)^{-1}\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\left(\mathbb{E}\left[(\zeta_2^Y - \zeta_1^Y)^2\mathbf{1}_{\{k,l\le\zeta_2^Y-\zeta_1^Y\}}\right]\mathbb{E}\left[(W_{z_k,l}+1)^4\right]\right)^{1/2}\\ \le C\mathbb{E}\left[(\zeta_2^Y - \zeta_1^Y)^4\right]^{1/4}\mathbb{E}\left[(W_{z_0,1}+1)^4\right]^{1/2}\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\mathbb{P}\left(k,l\le\zeta_2^Y-\zeta_1^Y\right)^{1/4}.\end{aligned}$$

Since the offspring distribution ξ^g has exponential moments we have that the time between regenerations has finite fourth moments by [65, Proposition 3]. That is, $\mathbb{E}\left[(\zeta_2^Y - \zeta_1^Y)^4\right] < \infty$.

Write Z_n and Z_n^g to be the GW-processes associated with $\mathcal{T}$ and $\mathcal{T}^g$. The number of excursions from the root is geometrically distributed with termination probability $1 - p_{ex}$ where

$$p_{ex} := \frac{Z_1 - Z_1^g}{Z_1}.$$

Using properties of geometric random variables we therefore have that

$$\mathbb{E}\left[(W_{z_0,1}+1)^4\right] \le C\mathbb{E}[(1-p_{ex})^{-4}] \le C\mathbb{E}[Z_1^4] < \infty$$

since $Z_1 \overset{\mathrm{d}}{=} \xi$ which has exponential moments.

It remains to show that

$$\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\mathbb{P}\left(k,l\le\zeta_2^Y-\zeta_1^Y\right)^{1/4} \tag{6.5}$$

is finite. Note that $\mathbb{P}\left(k,l\le\zeta_2^Y-\zeta_1^Y\right) = \mathbb{P}\left(\zeta_2^Y-\zeta_1^Y \ge l\right)$ whenever $l \ge k$. Using Chebyshev's inequality we can then bound (6.5) above by

$$2\sum_{k=1}^{\infty}\sum_{l=k}^{\infty}\mathbb{P}\left(\zeta_2^Y-\zeta_1^Y \ge l\right)^{1/4} \le 2\sum_{k=1}^{\infty}\sum_{l=k}^{\infty}\left(\frac{\mathbb{E}\left[\left(\zeta_2^Y-\zeta_1^Y\right)^j\right]}{l^j}\right)^{1/4}$$

for any integer j. In particular, by [65, Proposition 3] we have that $\mathbb{E}\left[\left(\zeta_2^Y-\zeta_1^Y\right)^j\right]$ is finite for any integer j. Choosing $j > 8$ we then have that this sum is finite which completes the proof. □

We now show that $\Sigma_{m_{tn}}$ approximates B_t^n. For $x \in \mathcal{T}$ let $\mathcal{T}_x$ denote the subtree consisting of all descendants of x in $\mathcal{T}$. Then, for $y \in \mathcal{T}^g$, let $\mathcal{T}_y^{*-}$ be the branch at y; that is, the subtree rooted at y consisting only of y, the children of y not on $\mathcal{T}^g$ and their descendants. The tree $\mathcal{T}_y^{*-}$ then has the law of a tree rooted at y with some random number $\mathcal{M}_y$ of h-GW-trees attached to y. Since $\mathcal{M}_y$ is dominated by ξ, by (6.1) we have that $\mathcal{M}_y$ has exponential moments. It therefore follows from [55, Theorem B] that there exists some constant C such that

$$\mathbf{P}(\mathcal{H}(\mathcal{T}_y^{*-}) \geq n) \leq Cf'(q)^n \tag{6.6}$$

where, for a fixed rooted tree $\mathscr{T}$, $\mathcal{H}(\mathscr{T}) := \sup\{d(\rho, x) : x \in \mathscr{T}\}$ is the height of $\mathscr{T}$. Let $\mathcal{H}_n := \max\{\mathcal{H}(\mathcal{T}_y^{*-}) : y \in \{Y_k\}_{k=0}^n\}$ denote the largest branch seen by Y by time n. It follows that

$$\sup_{t\in[0,T]} \left| B_t^n - \frac{\Sigma_{m_{tn}}}{\varsigma\sqrt{n}} \right| \leq \frac{|\varrho_1| + \nu_\beta\zeta_1^X + \mathcal{H}_{nT}}{\varsigma\sqrt{n}} + \sup_{j=1,\dots,m_{nT}} \frac{|\varrho_{j+1}| - |\varrho_j| + \nu_\beta(\zeta_{j+1}^X - \zeta_j^X)}{\varsigma\sqrt{n}}. \tag{6.7}$$

Up to time nT, the walk Y can have visited at most nT vertices on $\mathcal{T}^g$ therefore the probability that X has visited a branch of height at least $C\log(n)$ by time nT is at most $C_T nf'(q)^{C\log(n)}$. In particular, by the Borel-Cantelli lemma, choosing C suitably large we have that there are almost surely only finitely many n such that Y has visited the root of a branch of height at least $C\log(n)$ by time nT. Since $|\varrho_1|$ and ζ_1^X do not depend on n and have finite mean, we have that the first term in (6.7) converges $\mathbb{P}$-a.s. to 0.

By [65, Proposition 3], for any $k \in \mathbb{Z}^+$ we have that $\mathbb{E}[(|\varrho_2| - |\varrho_1|)^k] < \infty$ therefore the distance between regeneration points is small. In particular, bounding m_{nT} above by nT, using a union bound and Markov's inequality we have that for any $\varepsilon > 0$,

$$\mathbb{P}\left(\sup_{j=1,\dots,m_{nT}} \frac{|\varrho_{j+1}| - |\varrho_j|}{\varsigma\sqrt{n}} > \varepsilon\right) \leq C_{T,\varepsilon}\mathbb{E}\left[(|\varrho_2| - |\varrho_1|)^2 \mathbf{1}_{\{|\varrho_2|-|\varrho_1|>\varepsilon\sqrt{n}\}}\right]$$

which converges to 0 as $n \to \infty$ by dominated convergence. Similarly, using Proposition 6.1.2, we have that the same holds for the supremum of $\zeta_{j+1}^X - \zeta_j^X$; therefore, we have that

$$\mathbb{P}\left(\sup_{t\in[0,T]} \left| B_t^n - \frac{\Sigma_{m_{tn}}}{\varsigma\sqrt{n}} \right| > \varepsilon\right)$$

converges to 0 as $n \to \infty$.

By the law of large numbers and that ζ_1^X/n converges $\mathbb{P}$-a.s. to 0 we have that

$$\frac{\zeta_n^X}{n} = \frac{\zeta_1^X}{n} + \sum_{k=2}^n \frac{\zeta_k^X - \zeta_{k-1}^X}{n}$$

converges $\mathbb{P}$-a.s. It therefore follows by *continuity of the inverse at strictly increasing functions* (Proposition 2.1.1.v), that m_{nt}/n converges $\mathbb{P}$-a.s. to a deterministic linear process.

By Proposition 6.1.2 and the remark leading to it we have that Σ_m is the sum of i.i.d. centred random variables with finite second moment. By Donsker's invariance principle (see Section 2.3) we therefore have that $(\Sigma_{nt}/\sqrt{n})_{t\geq 0}$ converges to a scaled Brownian motion. By *continuity of composition at continuous limits* (Proposition 2.1.1.i), and the previous remarks we therefore have the following annealed central limit theorem.

Corollary 6.1.3. *Under the assumptions of Theorem 6.1, there exists a constant $\varsigma^2 > 0$ such that the process*

$$B_t^n := \frac{|X_{\lfloor nt \rfloor}| - n\nu_\beta t}{\varsigma\sqrt{n}}$$

converges in $\mathbb{P}$-distribution on $D_{J_1}([0,\infty),\mathbb{R})$ to a standard Brownian motion.

Remark 6.1.4. *The branch of a subcritical GW-tree conditioned to survive can be constructed by attaching a random number of subcritical GW-trees to a root vertex. In Lemma 3.2.6 we have shown that, conditional on having a single vertex in the first generation of the branch, the second moment of the first return time to the root is infinite whenever $\beta^2\tilde{\mu} \geq 1$ where $\tilde{\mu}$ is the mean of the subcritical GW-law. It therefore follows from this that*

$$\mathbf{E}\left[E_{\overline{\rho}}^{\overline{\mathcal{T}}^h}\left[(\tau_{\overline{\rho}}^+)^2\right]\right] = \infty$$

whenever $\beta^2 f'(q) \geq 1$ and $\mu > 1$. In particular, if we have that $\beta^2 f'(q) \geq 1$ then χ_j have infinite second moments. This strongly suggests that the condition $\beta^2 f'(q) < 1$ is necessary for the annealed central limit theorem however this remains to be proved. We also note here that when $p_0 = 0$ we have that $q = 0 = f'(q)$ and, therefore, this condition is necessarily satisfied.

We now extend Corollary 6.1.3 to a quenched functional central limit theorem. For each $n \in \mathbb{N}$ write $\mathbb{B}_t^n(X)$ to be the linear interpolation satisfying

$$\mathbb{B}_{k/n}^n(X) = \frac{|X_k| - k\nu_\beta}{\varsigma\sqrt{n}}$$

for $k \in \mathbb{N}$. We then have that $B_t^n = \mathbb{B}_t^n$ for $t > 0$ such that $nt \in \mathbb{N}$ and $|B_t^n - \mathbb{B}_t^n| \leq n^{-1/2}(\nu_\beta + 1)/\varsigma$ therefore it suffices to consider the interpolation. Lemma 6.1.5 yields a sufficient condition for proving Theorem 6.1 and follows from [21, Lemma 4.1] therefore we omit the proof.

Lemma 6.1.5. *Suppose that the assumptions of Theorem 6.1 hold and that for any bounded Lipschitz function $F : C([0,T],\mathbb{R}) \to \mathbb{R}$ and $b \in (1,2)$ we have that*

$$\sum_{k\geq 1} \mathrm{Var}_{\mathbf{P}}\left(E^{\mathcal{T}}\left[F\left(\mathbb{B}^{\lfloor b^k \rfloor}\right)\right]\right) < \infty. \tag{6.8}$$

Then the process $(B^n_t)_{t\geq 0}$ converges in $P^{\mathcal{T}}$-distribution on $D_{J_1}([0,\infty),\mathbb{R})$ to a standard Brownian motion for $\mathbf{P}$-a.e. $\mathcal{T}$.

We now complete the proof of the quenched functional CLT by following the method used in [65] to show that condition (6.8) holds for any bounded Lipschitz function $F : C([0,T],\mathbb{R}) \to \mathbb{R}$ and $b \in (1,2)$ under the assumptions of the theorem.

Proof of Theorem 6.1. For a fixed tree $\mathcal{T}$, let X^1, X^2 be independent β-biased walks on $\mathcal{T}$ and Y^1, Y^2 the corresponding embedded walks. For $i = 1,2,\ k \in \mathbb{N}$ and $t,s \geq 0$ let

$$\mathbb{B}^{k,i}_{t,s} = \mathbb{B}^{\lfloor b^k \rfloor}_t(X^i_{\cdot+s}) - \mathbb{B}^{\lfloor b^k \rfloor}_s(X^i_{\cdot})$$

be a random variable with law of the interpolation $\mathbb{B}^{\lfloor b^k \rfloor}$ started from the vertex X^i_s. Define

$$\vartheta^{Y^i}_k := \min\{m > \lfloor b^{k/4} \rfloor : m \in \{\zeta^{Y^i}_j\}_{j\geq 1}\} \quad \text{and} \quad \vartheta^{X^i}_k = \min\left\{m \geq 0 : X^i_m = Y^i_{\vartheta^{Y^i}_k}\right\}$$

to be the first regeneration time of Y^i after time $\lfloor b^{k/4} \rfloor$ and the corresponding time for X^i.

Let

$$\mathcal{A}^1_k := \left\{ Y^2_{\vartheta^{Y^2}_k} \notin \{Y^1_s :\ s \leq \vartheta^{Y^1}_k\} \right\} = \left\{ X^2_{\vartheta^{X^2}_k} \notin \{X^1_s :\ s \leq \vartheta^{X^1}_k\} \right\},$$
$$\mathcal{A}^2_k := \left\{ Y^1_{\vartheta^{Y^1}_k} \notin \{Y^2_s :\ s \leq \vartheta^{Y^2}_k\} \right\} = \left\{ X^1_{\vartheta^{X^1}_k} \notin \{X^2_s :\ s \leq \vartheta^{X^2}_k\} \right\}$$

and $\mathcal{A}_k := \mathcal{A}^1_k \cap \mathcal{A}^2_k$ be the event that, after the first regeneration times after time $\lfloor b^{k/4} \rfloor$, the paths of Y^1, Y^2 do not intersect. Write $\mathcal{B}^{k,i} := \{\vartheta^{Y^i}_k \leq b^{k/3}\}$ to be the event that the first regeneration after time $b^{k/4}$ happens before time $b^{k/3}$.

Recall that for $y \in \mathcal{T}^g$ we denote by $\mathcal{H}(\mathcal{T}^{*-}_y)$ the height of the branch attached to the vertex y. Using Lipschitz properties of $\mathbb{B}^{k,i}$ we have that

$$\begin{aligned}
&\sup_{t\leq T}\left|\mathbb{B}^{k,i}_{t,0} - \mathbb{B}^{k,i}_{t,\vartheta^{X^i}_k}\right| \\
&\quad\leq \sup_{m\leq Tb^k} b^{-k/2}\left||X^i_m| - m\nu_\beta - |X^i_{m+\vartheta^{X^i}_k}| + (m+\vartheta^{X^i}_k)\nu_\beta + |X^i_{\vartheta^{X^i}_k}| - \vartheta^{X^i}_k\nu_\beta\right| \\
&\quad= \sup_{m\leq Tb^k} b^{-k/2}\left||X^i_m| - |X^i_{m+\vartheta^{X^i}_k}| + |X^i_{\vartheta^{X^i}_k}|\right| \\
&\quad\leq b^{-k/2}\max_{m\leq Tb^k}\left|\left||Y^i_m| - |Y^i_{m+\vartheta^{Y^i}_k}|\right| + |Y^i_{\vartheta^{Y^i}_k}|\right| + b^{-k/2}\mathcal{H}^i_{Tb^k}
\end{aligned}$$

where $\mathcal{H}^i_{Tb^k}$ is the height of the tallest branch seen by time Tb^k by Y^i. By time Tb^k the walk Y^i can visit at most $Tb^k + 1$ unique vertices. At the first hitting time of a vertex, the bud and backbone distribution from this vertex are independent of the past; therefore, by (6.6)

$$\begin{aligned}\mathbb{P}\left(\mathcal{H}^i_{Tb^k} \geq C\log(b^k)\right) &\leq C_T b^k \mathbf{P}(\mathcal{H}(\mathcal{T}^{*-}_\rho) \geq C\log(b^k)) \\ &\leq C_T b^k f'(q)^{C\log(b^k)} \\ &\leq C_T b^{-k} \end{aligned} \tag{6.9}$$

for C sufficiently large. Furthermore, by the Lipschitz property of Y^i we have that

$$b^{-k/2}\max_{m\leq Tb^k}\left|\left||Y^i_m| - |Y^i_{m+\vartheta^{Y^i}_k}|\right| + |Y^i_{\vartheta^{Y^i}_k}|\right| \leq 2\vartheta^{Y^i}_k b^{-k/2}$$

which is bounded above by $2b^{-k/6}$ on the event $\mathcal{B}^{k,i}$. Letting $\mathcal{C}^{k,i} := \{\mathcal{H}^i_{Tb^k} < C\log(b^k)\}$, we then have that, on the event $\mathcal{B}^{k,i}\cap\mathcal{C}^{k,i}$,

$$\left|F\left(\mathbb{B}^{k,i}_{\cdot,0}\right) - F\left(\mathbb{B}^{k,i}_{\cdot,\vartheta^{X^i}_k}\right)\right| \leq Cb^{-k/6}$$

for any bounded Lipschitz function $F : C([0,T],\mathbb{R}) \to \mathbb{R}$.

Using the Lipschitz and boundedness properties of F, we then have that

$$\begin{aligned}&\mathrm{Var}_{\mathbf{P}}\left(E^{\mathcal{T}}\left[F\left(\mathbb{B}^{\lfloor b^k\rfloor}\right)\right]\right) \\ &= \mathbf{E}\left[E^{\mathcal{T}}\left[F(\mathbb{B}^{\lfloor b^k\rfloor})\right]^2\right] - \mathbf{E}\left[E^{\mathcal{T}}\left[F(\mathbb{B}^{\lfloor b^k\rfloor})\right]\right]^2 \\ &= \mathbb{E}\left[F(\mathbb{B}^{k,1})F(\mathbb{B}^{k,2})\right] - \mathbb{E}\left[F(\mathbb{B}^{k,1})\right]\mathbb{E}\left[F(\mathbb{B}^{k,2})\right] \\ &\leq C\left(\mathbb{P}\left((\mathcal{B}^{k,1})^c\right) + \mathbb{P}\left((\mathcal{C}^{k,1})^c\right) + b^{-k/6}\right) \\ &\qquad + \mathbb{E}\left[F\left(\mathbb{B}^{k,1}_{\cdot,\vartheta^{X^1}_k}\right)F\left(\mathbb{B}^{k,2}_{\cdot,\vartheta^{X^2}_k}\right)\right] - \mathbb{E}\left[F\left(\mathbb{B}^{k,1}_{\cdot,\vartheta^{X^1}_k}\right)\right]\mathbb{E}\left[F\left(\mathbb{B}^{k,2}_{\cdot,\vartheta^{X^2}_k}\right)\right].\end{aligned}$$

On the event $\mathcal{A}_k$ we have that $\mathbb{B}^{k,1}_{\cdot,\vartheta^{X^1}_k}$, $\mathbb{B}^{k,2}_{\cdot,\vartheta^{X^2}_k}$ are independent therefore

$$\mathbb{E}\left[F\left(\mathbb{B}^{k,1}_{\cdot,\vartheta^{X^1}_k}\right)F\left(\mathbb{B}^{k,2}_{\cdot,\vartheta^{X^2}_k}\right)|\mathcal{A}_k\right] - \mathbb{E}\left[F\left(\mathbb{B}^{k,1}_{\cdot,\vartheta^{X^1}_k}\right)|\mathcal{A}_k\right]\mathbb{E}\left[F\left(\mathbb{B}^{k,2}_{\cdot,\vartheta^{X^2}_k}\right)|\mathcal{A}_k\right] = 0.$$

Using the Lipschitz property of F we then have that

$$\mathrm{Var}_{\mathbf{P}}\left(E^{\mathcal{T}}\left[F\left(\mathbb{B}^{\lfloor b^k\rfloor}\right)\right]\right) \leq C\left(\mathbb{P}\left((\mathcal{A}^{k,1})^c\right) + \mathbb{P}\left((\mathcal{B}^{k,1})^c\right) + \mathbb{P}\left((\mathcal{C}^{k,1})^c\right) + b^{-k/6}\right).$$

For $i = 1,2$ we have that Y^i are biased random walks on a supercritical GW-tree without leaves $\mathcal{T}^g$, whose offspring law has exponential moments. It follows that the estimates $\mathbb{P}((\mathcal{A}^{k,1})^c), \mathbb{P}((\mathcal{B}^{k,1})^c) \leq b^{-\tilde{c}k}$ given in the proof of [65, Theorem 3] still hold. Combining these with (6.9) we have that there exists $c > 0$ such that for k

sufficiently large

$$\mathrm{Var}_{\mathbf{P}}\left(E^{\mathcal{T}}\left[F\left(\mathbb{B}^{\lfloor b^k\rfloor}\right)\right]\right) \leq Cb^{-ck}$$

which shows (6.8) and therefore the result follows from Lemma 6.1.5.

□

6.2 Sub-ballistic regimes

In the section we study the biased random walk on the supercritical GW-tree conditioned to survive in the sub-ballistic regime ($\beta > f'(q)^{-1}$). Recall from (1.2) that $\gamma = -\log(f'(q))/\log(\beta)$. The aim of this section is to prove Theorem 6.2 which states that if ξ belongs to the domain of attraction of a stable law with index $\alpha \in (1,2)$ then $\Delta_n n^{-1/\gamma}$ converges in distribution along subsequences $n_l(t) = \lfloor tf'(q)^{-l}\rfloor$ for $t > 0$, that the laws of $(\Delta_n n^{-1/\gamma})_{n\geq 0}$ and $(|X_n|n^{-\gamma})_{n\geq 0}$ under $\mathbb{P}$ are tight on $(0,\infty)$ and $\mathbb{P}$-a.s.

$$\lim_{n\to\infty}\frac{\log|X_n|}{\log(n)} = \gamma.$$

In [10] it is shown that this holds when $\mathbf{E}[\xi^2] < \infty$ with the same conditions on the mean and bias: $\mu > 1$, $\beta > f'(q)^{-1}$. In order to extend this result to prove Theorem 6.2 it will suffice to prove Lemmas 6.2.1, 6.2.2 and 6.2.3 which we defer to the end of the section following a breakdown of the proof.

Recall that $\mathcal{T}$ denotes the supercritical tree conditioned to survive, $\mathcal{T}^f$ the unconditioned tree and $\mathcal{T}^h$ the tree conditioned to die out. Write Z_n, Z_n^f, Z_n^h to be the size of their n^{th} generations respectively and V_n, V_n^f to be the number of vertices in the n^{th} generation of the backbone (for $\mathcal{T}$ and $\mathcal{T}^f$) where we note that V_n^f can be 0 if the tree is finite. As in the subcritical case we define $\mathcal{T}^{*-}$ to be a dummy branch formed by a backbone vertex, its buds and the associated traps.

In Figure 6.1, the dashed lines represent the finite structures comprised of the buds and leaves. It will be convenient to refer to the traps at a site so for $x \in \mathcal{Y}$ let L_x denote the collection of traps adjacent to x, for example in Figure 6.1 L_ρ consists of the two trees rooted at y, z. We then write $\mathcal{T}_x^{*-}$ to be the branch at x.

In Lemma 6.2.1 we show that $\mathbf{P}(\mathcal{H}(\mathcal{T}^{*-}) > n) \sim C^* f'(q)^n$ for some constant C^*. For the supercritical tree this is the same as when $\mathbf{E}[\xi^2] < \infty$ whereas for the subcritical tree the exponent changes depending on the stability. This holds because the first moment of the bud distribution has a fundamental role and the change from finite to infinite variance changes this for the subcritical tree but not for the supercritical tree. Lemma 6.2.1 is an extension of [10, Lemma 6.1] which is proved using a Taylor expansion of the f around 1 up to second moments. We cannot take this approach because $f''(1) = \infty$; instead we use the form of the generating function determined in

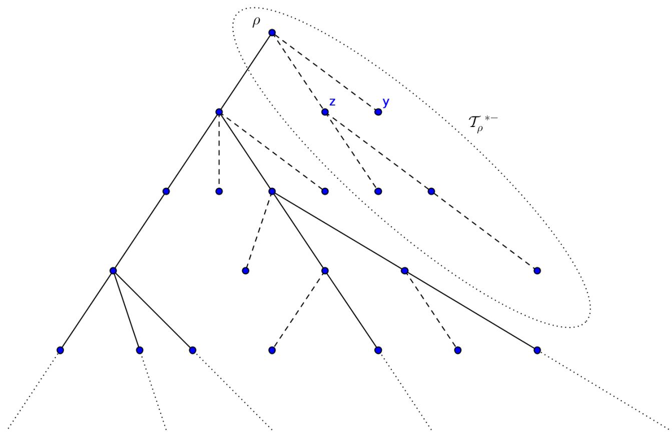

Figure 6.1: A sample section of a supercritical tree with solid lines representing the backbone and dashed lines representing the dangling ends.

Lemma 2.2.2. The expression is important because the expected time spent in a large branch of height $\mathcal{H}(\mathcal{T}^{*-})$ is approximately $c\beta^{\mathcal{H}(\mathcal{T}^{*-})}$ for some constant c therefore this is the key ingredient in determining the correct scaling.

Lemma 6.2.2 shows that, with high probability, no large branch contains more than one large trap. This is important because the number of large traps would affect the escape probability and a more detailed decomposition of the excursion times would be required. That is, if there are many large traps in a branch then it is likely that the root has many offspring on the backbone since a geometric number of the offspring lie on the backbone. The analogue of this in [10] is proved using the bound $f'(1) - f'(1-\epsilon) \leq C\epsilon$ which follows because $f''(1) < \infty$. Similarly to Lemma 6.2.1, we use a more precise form of f in order to obtain a similar bound.

Lemma 6.2.3 shows that no branch visited by level n is too large. This is important for the tightness result since we need to bound the deviation of the walk from the furthest point reached along the backbone. The proof of this follows quite straightforwardly from Lemma 6.2.1.

To explain why these are needed, we recall the argument which follows a similar structure to the proof of Theorem 4.2. We say that a branch is large if its height exceeds a certain threshold $h_{n,\varepsilon}$ which we define later. The first part of the argument involves showing that, asymptotically, the time spent outside large branches is negligible. This follows by the techniques used for the subcritical tree. That is, we use several crude union bounds and the formula (2.7) for the expected time spent in a tree.

One of the major difficulties with the walk on the supercritical tree is determining the distribution over the number of entrances into a large branch. The height of the branch from a backbone vertex x will be correlated with the number of children x has on the backbone. This changes the escape probability and therefore the number of excursions into the branch. It can be shown that the number of excursions into the first large trap converges in distribution to some non-trivial random variable W_∞. In particular, it is shown in [10] that W_∞ can be stochastically dominated by a geometric random variable and that there is a constant $c_W > 0$ such that $\mathbb{P}(W_\infty > 0) \geq c_W$. By Lemma 6.2.2 we have that large branches have only one bud which is the root of a large trap. The convergence to W_∞ therefore says that the number of entrances into this unique large is independent of the height.

It can be shown that, asymptotically, the large branches are independent in the sense that, with high probability, the walk will not reach one large branch and then return to a previously visited large branch. Using Lemmas 6.2.1 and 6.2.2 (among other results) it can then be shown that Δ_n can be approximated by the sum of i.i.d. random variables.

The remainder of the proof of the first part of Theorem 6.2 involves decomposing the time spent in large branches, showing that the suitably scaled excursion times converge in distribution and proving convergence results for sums of i.i.d. variables. Since $\mathbf{P}(Z_1^h = k) = p_k q^{k-1}$, the subcritical GW law over the traps has exponential moments. This means that these final parts of the proof follow by the results proven in [10] since, by Lemma 6.2.1, the scaling is the same as when $\mathbf{E}[\xi^2] < \infty$.

Tightness of the sequences $(\Delta_n n^{-1/\gamma})_{n\geq 0}$, $(X_n n^{-\gamma})_{n\geq 0}$ and almost sure convergence of $\log(|X_n|)/\log(n)$ then follow by the proof of [10, Theorem 1.1] (with one slight adjustment) which is similar to the proof of Theorem 4.4. In order to bound the maximum distance between the walker's current position and the last regeneration point we use a bound on the maximum height of a trap seen up to Δ_n^Y. In [10] it is shown that the probability a trap of height at least $4\log(n)/\log(f'(q)^{-1})$ is seen is at most order n^{-2} by using finite variance of the offspring distribution to bound the variance of the number of traps in a branch. In Lemma 6.2.3 we prove this using Lemma 6.2.1.

Lemma 6.2.1. *Under the assumptions of Theorem 6.2*

$$\mathbf{P}(\mathcal{H}(\mathcal{T}^{*-}) > n) \sim C^* f'(q)^n$$

where $C^* = q(\mu - f'(q))c_\mu/(1-q)$ *and* c_μ *is such that* $\mathbf{P}(\mathcal{H}(\mathcal{T}^h) \geq n) \sim c_\mu f'(q)^n$.

Proof. Let $Z := Z_1$, $Z^f := Z_1^f$, $V := V_1$, $V^f := V_1^f$ and $s_n := \mathbf{P}(\mathcal{H}(\mathcal{T}^h) < n)$, then denote by $(\mathcal{H}_k)_{k\geq 1}$ independent random variables with the distribution of $\mathcal{H}(\mathcal{T}^h)$. We then have that

$$\begin{aligned}\mathbf{P}(\mathcal{H}(\mathcal{T}^{*-}) > n) &= 1 - \frac{\mathbf{P}(\max_{k=1,\dots,Z^f - V^f} \mathcal{H}_k < n, V^f > 0)}{\mathbf{P}(V^f > 0)} \\ &= 1 - \frac{\mathbf{E}\left[\mathbf{P}(\max_{k=1,\dots,Z^f-V^f} \mathcal{H}_k < n, V^f > 0 | Z^f, V^f)\right]}{1-q}\end{aligned}$$

$$= 1 - \frac{\mathbf{E}\left[s_n^{Z^f - V^f}\mathbf{1}_{\{V^f>0\}}\right]}{1-q}$$
$$= 1 - \frac{\mathbf{E}\left[s_n^{Z^f - V^f}1^{V^f}\right] - \mathbf{E}\left[s_n^{Z^f}\mathbf{1}_{\{V^f=0\}}\right]}{1-q}.$$

For any $t, s > 0$

$$\begin{aligned}
\mathbf{E}\left[s^{Z^f - V^f}t^{V^f}\right] &= \mathbf{E}\left[s^{Z^f}\mathbf{E}[(t/s)^{V^f}|Z]\right] \\
&= \mathbf{E}\left[s^{Z^f}\sum_{k=0}^{Z^f}(t/s)^k\binom{Z^f}{k}q^{Z^f-k}(1-q)^k\right] \\
&= \mathbf{E}\left[(qs)^{Z^f}\left(1+\frac{t(1-q)}{qs}\right)^{Z^f}\right] \\
&= f(sq + t(1-q)).
\end{aligned}$$

Furthermore,

$$\mathbf{E}\left[s^{Z^f}\mathbf{1}_{\{V^f=0\}}\right] = \mathbf{E}\left[s^{Z^f}\mathbf{P}(V^f=0|Z^f)\right] = \mathbf{E}\left[(sq)^{Z^f}\right] = f(sq).$$

Therefore, writing $t_n := s_nq + 1 - q$ we have that $1 - t_n = q(1-s_n)$ and

$$\mathbf{P}(\mathcal{H}(\mathcal{T}^{*-}) > n) = 1 - \frac{f(s_nq+1-q)}{1-q} + \frac{f(s_nq)}{1-q} = \frac{(1-f(t_n)) - (q - f(s_nq))}{1-q}.$$

By Taylor we have that $\exists z \in [s_nq, q]$ such that

$$f(s_nq) = q + qf'(q)(s_n-1) + \frac{f''(z)q^2(s_n-1)^2}{2}.$$

Since $q < 1$ we have that $f''(z)$ exists for all $z \le q$ and is bounded above by $f''(q) < \infty$.

By Lemma 2.2.2

$$1 - f(t_n) = \mu(1-t_n) + \frac{\Gamma(3-\alpha)}{\alpha(\alpha-1)}(1-t_n)^\alpha\overline{L}\left((1-t_n)^{-1}\right)$$

for a slowly varying function $\overline{L}$. In particular,

$$\begin{aligned}
\frac{\mathbf{P}(\mathcal{H}(\mathcal{T}^{*-}) > n)}{(1-s_n)} &= \frac{q}{1-q}\left(\frac{1-f(t_n)}{1-t_n} - f'(q) + f''(z)q(s_n-1)/2\right) \\
&= \frac{q}{1-q}\left(\mu - f'(q) + O\left((1-s_n)^{\alpha-1}\overline{L}\left((1-s_n)^{-1}\right)\right)\right) \qquad (6.10) \\
&\sim \frac{q(\mu - f'(q))}{1-q}
\end{aligned}$$

which is the desired result. $\square$

Recall that Δ_n^Y is the first hitting time of level n of the backbone by Y_n and L_x is the collection of traps adjacent to x then for $\varepsilon > 0$ let

$$h_{n,\varepsilon} := \left\lceil \frac{(1-\varepsilon)\log(n)}{\log(f'(q)^{-1})} \right\rceil \qquad \text{and} \qquad B(n) := \bigcap_{i=0}^{\Delta_n^Y} \{|\{\mathscr{T} \in L_{Y_i} : \mathcal{H}(\mathscr{T}) \geq h_{n,\varepsilon}\}| \leq 1\}$$

denote the critical height of a trap and the event that any backbone vertex seen up to reaching level n has at most one $h_{n,\varepsilon}$-trap (which is $C_3(n)$ of [10]) respectively.

Lemma 6.2.2. *Under the assumptions of Theorem 6.2 we have that* $\lim_{n\to\infty} \mathbb{P}(B(n)^c) = 0$.

Proof. Using [10, Lemma 7.2] we have that the number of backbone vertices visited by Δ_n is at most Cn with high probability therefore

$$\mathbb{P}(B(n)^c) \leq o(1) + Cn\left(\mathbf{P}(\mathcal{H}(\mathcal{T}^{*-}) > h_{n,\varepsilon}) - \mathbf{P}(|\{\mathscr{T} \in L_\rho : \mathcal{H}(\mathscr{T}) \geq h_{n,\varepsilon}\}| = 1)\right).$$

Recall $s_{h_{n,\varepsilon}} = \mathbf{P}(\mathcal{H}(\mathcal{T}^h) < h_{n,\varepsilon})$ and from (6.10) we have that

$$\mathbf{P}(\mathcal{H}(\mathcal{T}^{*-}) > h_{n,\varepsilon}) = (1-s_{h_{n,\varepsilon}})\frac{q(\mu - f'(q))}{1-q} + O((1-s_{h_{n,\varepsilon}})^\alpha \overline{L}((1-s_{h_{n,\varepsilon}})^{-1})$$

for some slowly varying function $\overline{L}$.

Similarly to the method used in Lemma 6.2.1 we have that

$$\begin{aligned}
&\mathbf{P}(|\{\mathscr{T} \in L_\rho : \mathcal{H}(\mathscr{T}) \geq h_{n,\varepsilon}\}| = 1) \\
&\qquad = \sum_{k=1}^\infty \sum_{j=1}^{k-1} \mathbf{P}\left(Z_1 = k, V_1 = k-j, |\{\mathscr{T} \in L_\rho : \mathcal{H}(\mathscr{T}) \geq h_{n,\varepsilon}\}| = 1\right) \\
&\qquad = \sum_{k=1}^\infty \mathbf{P}(Z_1 = k) \sum_{j=1}^{k-1} \mathbf{P}(V_1 = k-j | Z_1 = k) j (1-s_{h_{n,\varepsilon}}) s_{h_{n,\varepsilon}}^{j-1} \\
&\qquad = \sum_{k=1}^\infty \frac{(1-q^k)p_k}{1-q} \sum_{j=1}^{k-1} \binom{k}{j} \frac{q^j(1-q)^{k-j}}{1-q^k} j (1-s_{h_{n,\varepsilon}}) s_{h_{n,\varepsilon}}^{j-1} \\
&\qquad = \frac{q(1-s_{h_{n,\varepsilon}})}{1-q} \sum_{k=1}^\infty kp_k \sum_{j=0}^{k-2} \frac{(k-1)!(qs_{h_{n,\varepsilon}})^j(1-q)^{k-1-j}}{j!(k-1-j)!} \\
&\qquad = \frac{q(1-s_{h_{n,\varepsilon}})}{1-q} \sum_{k=1}^\infty kp_k \left((qs_{h_{n,\varepsilon}} + 1 - q)^{k-1} - (qs_{h_{n,\varepsilon}})^{k-1}\right) \\
&\qquad = \frac{q(1-s_{h_{n,\varepsilon}})}{1-q} \left(f'(t_{h_{n,\varepsilon}}) - f'(qs_{h_{n,\varepsilon}})\right)
\end{aligned}$$

where $t_{h_{n,\varepsilon}} = qs_{h_{n,\varepsilon}} + 1 - q$. By Taylor $f'(qs_n) = f'(q) + O(1 - s_{h_{n,\varepsilon}})$ as $n \to \infty$. From Lemma 2.2.2 we have that $1 - f(t_{h_{n,\varepsilon}}) = \mu(1 - t_{h_{n,\varepsilon}}) + (1 - t_{h_{n,\varepsilon}})^\alpha \overline{L}((1 - t_{h_{n,\varepsilon}})^{-1})$ for some slowly varying function $\overline{L}$. Applying [53, Theorem 2] we have that $f'(t_{h_{n,\varepsilon}}) = \mu + O((1 - t_{h_{n,\varepsilon}})^{\alpha-1}\overline{L}((1 - t_{h_{n,\varepsilon}})^{-1}))$. In particular,

$$\begin{aligned}\mathbf{P}\Big(\big|\{ \mathscr{T} \in L_\rho : \mathcal{H}(\mathscr{T}) \geq h_{n,\varepsilon}\}\big| = 1\Big) \\ = \frac{q(1 - s_{h_{n,\varepsilon}})}{1 - q}\Big(\mu - f'(q) + O\left((1 - t_{h_{n,\varepsilon}})^{\alpha-1}\overline{L}\big((1 - t_{h_{n,\varepsilon}})^{-1}\big)\right)\Big)\end{aligned}$$

since $\alpha < 2$, thus

$$\mathbb{P}(B(n)^c) \leq o(1) + O(n(1 - t_{h_{n,\varepsilon}})^\alpha \overline{L}((1 - t_{h_{n,\varepsilon}})^{-1})).$$

There exists some constant c such that $1 - t_{h_{n,\varepsilon}} \sim qc_\mu f'(q)^{h_{n,\varepsilon}} \leq cn^{-(1-\varepsilon)}$ therefore since $\alpha > 1$ we can choose $\varepsilon > 0$ small enough (depending on α) such that $\mathbb{P}(B(n)^c)$ converges to 0 as $n \to \infty$. $\square$

Let $D(n) := \Big\{\max_{j \leq \Delta_n^Y} \mathcal{H}(\mathcal{T}_{Y_j}^{*-}) \leq 3\log(n)/\log(f'(q)^{-1})\Big\}$ be the event that all branches seen before reaching level n are of height at most $4\log(n)/\log(f'(q)^{-1})$.

Lemma 6.2.3. *Under the assumptions of Theorem 6.2 we have that* $\mathbb{P}\left(D(n)^c\right) \leq Cn^{-2}$ *for some constant* C.

Proof. By comparison with a biased random walk on $\mathbb{Z}$, standard large deviations estimates yield that for C sufficiently large $\mathbb{P}(\Delta_n^Y > Cn) \leq cn^{-2}$ for some constant c. Using Lemma 6.2.1 we then have that for independent branches $\mathcal{T}_j^{*-}$

$$\begin{aligned}\mathbf{P}\left(\bigcup_{j=1}^{cn} \mathcal{H}(\mathcal{T}_j^{*-}) > \frac{3\log(n)}{\log(f'(q)^{-1})}\right) &\leq cn\mathbf{P}\left(\mathcal{H}(\mathcal{T}^{*-}) > \frac{3\log(n)}{\log(f'(q)^{-1})}\right) \\ &\leq Cnf'(q)^{\frac{3\log(n)}{\log(f'(q)^{-1})}} \\ &= Cn^{-2}.\end{aligned}$$

$\square$

As explained at the beginning of the section, these three results, combined with [10, Theorems 1.1 & 1.3], are sufficient to complete the proof of Theorem 6.2.

Glossary

The notation used in this thesis is vast and, although an attempt at consistency has been made, there is some notation reused between chapters for different purposes. Included here is a list of some of the most important and frequently used notation as well as a note on several duplications.

To begin, the notation stated in the introduction will remain reasonably consistent throughout with several minor exceptions. Firstly, we use X for both the randomly trapped random walk and the random walk on a GW-tree interchangeably since, in context, it should be clear to which walk we are referring. The same goes for the embedded walk Y. The stability index α is used for both the stability of the holding times in the randomly trapped random walk model and also the stability of the offspring law in the GW-tree model. The functions f, g and h are introduced as probability generating functions of GW-trees however we will often use them as arbitrary functions. We introduce χ_j as independent Bernoulli random variables however they are later used for other variables. Similarly, for the GW-tree model Z_n denotes a GW-process whereas Z is often used to denote other random variables. Finally, the notation $G(V, E)$ for a graph is never used after the introduction and we often use G to denote a geometric random variable.

X_t	trapped walk
Y_n	embedded walk
β	bias of the walk
α	stability index of heavy tailed variables
γ	a scaling exponent for the walk
τ_x^+	first return time to x
v_x	number of visits to x up to a given time
ω	environment
$\eta = (\eta_k)_{k\geq 0}$	sequence of holding times
S_n	clock process
$\mathcal{L}$	local times of Y
μ	mean of the offspring distribution
σ^2	variance of the offspring distribution

$\mathscr{T}$	fixed tree
$\mathscr{T}_x$	the descendent tree of x in $\mathscr{T}$
ρ	root of $\mathscr{T}$
$d(x, y)$	graph distance between points x, y
$\lvert x \rvert := d(\rho, x)$	graph distance between x and the root of the tree
$\mathcal{H}(\mathscr{T})$	height of $\mathscr{T}$
$c(x)$	children of x in $\mathscr{T}$
$\overleftarrow{x}$	parent of x in $\mathscr{T}$
ξ	offspring variable
ξ^*	size-biased variable
$\mathcal{T}$	GW-tree conditioned to survive
$\mathcal{Y}$	backbone of $\mathcal{T}$
$\mathcal{T}^f$	GW-tree with the p.g.f. f
ξ^f	offspring variable with p.g.f. f
$\mathcal{T}^{*-}$	a branch of $\mathcal{T}$
ν_β	speed of the walk
B_t^n	Brownian approximation
$\mathbb{B}$	interpolation of B_t^n
ζ_m^Y, ζ_t^X	regeneration times for the walks Y, X
ϱ_k	regeneration points for the walk Y
m_t	number of regenerations by time t
Δ_n	first hitting time of level n
P	law over the embedded walk Y
P^ω	quenched law
$\mathbf{P}$	environment law
$\mathbb{P}$	annealed law
$D([0,\infty), \mathbb{R})$	càdlàg functions $f : [0,\infty) \to \mathbb{R}$
$D_V([0,\infty), \mathbb{R})$	$D([0,\infty), \mathbb{R})$ equipped with topology V
a_n	scaling sequence of heavy tailed variables
L, $\tilde{L}$	functions which vary slowly at ∞

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